Sequential Continuity

Theorem 1. *Let $S \subset \mathbb{R}^n$. Let $a \in S$ and let $f : S \to \mathbb{R}^m$. Then f is continuous at a if and only if $f(x_n) \to f(a)$ for all sequences $x_n \in S,\ x_n \to a$.*

Proof. First suppose f is continuous at a. Let $x_n \in S,\ x_n \to a$. Let $\epsilon > 0$ be given. Chose $\delta > 0$ so that if $||x - a|| < \delta$ then $||f(x) - f(a)|| < \epsilon$. Now choose N so that if $n > N$ then $||x_n - a|| < \delta$. Then $||f(x_n) - f(a)|| < \delta$, so $f(x_n) \to f(a)$. Notice this is correct even when a is an isolated point of S.

Next suppose f is not continuous at a. If f is not continuous at a then a cannot be an isolated point, since every function is continuous at an isolated point of its domain. If f is not continuous there is some ϵ for which no matter how what δ we choose there is a point $x_n \in S$ with $||f(x_n) - f(a)|| \geq \epsilon$. So let's take $\delta = 1/n$ and $x_n \in S,\ ||x_n - a|| < 1/n, ||f(x_n) - f(a)|| \geq \epsilon$. Then $x_n \to a$ but $f(x_n) \not\to f(a)$. Hence some sequence of points x_n converges to a but $f(x_n)$ does not converge to $f(a)$

Notice the equivalence does not require proof at at isolated point, since every function is continuous at an isolated point and every sequence x_n that converges to an isolated point satisfies $x_n = a$ for large enough n. □