

# Sample Problems

Math 334

The final exam will be held from **8:30 - 10:20 a.m.** on **Monday, December 14**. You may bring one notebook size sheet of paper with notes on *both* sides. There may be homework or example problems on the final exam, in addition to problems similar to the problems on this sheet and the previous sample problem sheets. This is quite a long list of problems. You should do as many as you can. Also you should be prepared to define, state, or use the terms and theorems at the end of this sheet. The final will be comprehensive and will cover through §5.2 in Folland.

1. (a) Compute  $\int_{x^2+y^2=1} \frac{-ydx + xdy}{x^2 + y^2}$ .

(b) Using part (a) and Green's theorem, compute  $\int_{\frac{x^2}{4} + \frac{y^2}{9} = 1} \frac{-ydx + xdy}{x^2 + y^2}$ .

2. Show that the sequence  $\sqrt{n^2 + n} - n$  converges and find its limit.

3. Give an example of a function  $f : [0, 1] \rightarrow \mathbb{R}$  defined (and finite) at all points of  $[0, 1]$  such that  $f$  is not bounded on any subinterval of  $[0, 1]$ .

4. Suppose  $f \in C^2(\mathbb{R})$  and  $f''(t) \geq 0$ . Prove that the set of critical points of  $f$  is a interval.

5. Let  $P(x)$  be the parallelogram with vertices

$$(0, 0), (f(x), f'(x)), (g(x), g'(x)), (f(x) + g(x), f'(x) + g'(x))$$

where  $f'' = qf, g'' = qg$  and  $q(x)$  is some continuous function. Let  $A(x)$  be the area of this parallelogram. Show that  $A(x)$  is constant.

6. Let  $I$  be an interval in  $\mathbb{R}$ .  $I$  might be open, closed, or neither. Let  $f$  be a real valued continuous function defined on  $I$ . Suppose  $f$  has no local maxima or minima in the interior of  $\mathbb{R}$ . Then prove that  $f$  is monotonic.

7. Let  $0 < a < b$  and  $K > 0$ . Prove that

$$\left| \int_a^b \frac{\sin(Kt)}{t} dt \right| \leq \pi.$$

8. Let  $I$  be an interval in  $\mathbb{R}$ .  $I$  might be open, closed, or neither. Suppose  $f : I \rightarrow \mathbb{R}$  is strictly increasing. Prove that if the image of  $f$  is connected then  $f$  is continuous.

9. Let  $f(x, y)$  be defined for  $0 \leq x \leq 1, 0 \leq y \leq 1$  by

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 2y & \text{if } x \text{ is rational.} \end{cases}$$

- (a) Prove that  $\int_0^1 \left( \int_0^1 f(x, y) dy \right) dx = 1$ .  
 (b) What can you say about  $\int_0^1 \left( \int_0^1 f(x, y) dx \right) dy$ ?  
 (c) Is  $f$  integrable?

10. Suppose  $a < b < c < d$ . Let  $I = [a, b]$ ,  $J = [c, d]$ ,  $R = I \times J$  and let  $f(x, y) = |x - y|$ , if  $x \in I$ ,  $y \in J$ . Compute  $\int_R f$ .

11. Find the volume of the set

$$\left\{ \left( \frac{x}{1-z} \right)^2 + \left( \frac{y}{1+z} \right)^2 < 1, -1 < z < 1 \right\}$$

12. Let  $f$  be a  $C^1$  real-valued function on  $\mathbf{R}^1$  and define a transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  by the formulas  $u = f(x), v = -y + xf(x)$ . Suppose that  $f'(x_0) \neq 0$ . Show that this transformation is invertible near  $(x_0, y_0)$  for any  $y_0$ . Show that the inverse has the form  $x = g(u), y = -v + ug(u)$  for some  $C^1$  function  $g$ , defined near  $f(x_0)$ .
13. Find the volume of the solid bounded by the  $xy$ -plane, the cylinder  $\{(x, y, z) : x^2 + y^2 = 2x\}$ , and the cone  $\{(x, y, z) : z = \sqrt{x^2 + y^2}\}$ .
14. Let  $\Pi$  be the parallelotope in  $\mathbf{R}^3$  spanned by vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . Let  $\theta_1, \theta_2, \theta_3$  be the angles between  $\mathbf{v}_1, \mathbf{v}_2; \mathbf{v}_1, \mathbf{v}_3; \mathbf{v}_2, \mathbf{v}_3$ . Prove that the volume of  $\Pi$  is the square root of

$$|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 |\mathbf{v}_3|^2 (1 + 2 \cos \theta_1 \cos \theta_2 \cos \theta_3 - (\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3)).$$

15. Let  $S = \{(x, y, z) : a \leq x \leq y \leq z \leq b\}$ . Prove that

$$\int_S f(x)f(y)f(z) dx dy dz = \frac{1}{6} \left( \int_a^b f \right)^3$$

16. Let  $f$  be continuous on  $\mathbb{R}$  and suppose  $f(x+1) = f(x)$  for all  $x \in \mathbb{R}$ . Suppose  $a > 0$ . Prove that

$$\int_0^1 (f(x+a) - f(x)) dx = 0.$$

Use this to prove that there exist  $x_1, x_2$  with  $0 \leq x_1 < x_2 < 1$  so that

$$\begin{aligned} f(x_1 + a) &= f(x_1) \\ f(x_2 + a) &= f(x_2). \end{aligned}$$

17. (a) Let  $C_n$  be the curve  $\{(x, \sin \frac{1}{x}) : \frac{1}{n} \leq x \leq 1\}$ . Prove that the length  $L(C_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .  
(b) Prove that the area of the set  $\cup_{n=1}^{\infty} C_n$  is 0.

18. Let  $u$  be a function defined on  $\mathbf{R}^n$  which is homogeneous of degree  $k$ . Prove that  $\nabla^2 u$  is homogeneous of degree  $k - 2$ . Let  $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\mathbf{x}|$ . Compute  $\nabla^2 r^k$ .

19. Compute the area in the first quadrant between the four curves

$$x^3 = a^2 y, x^3 = b^2 y, y^3 = \alpha^2 x, y^3 = \beta^2 x,$$

where  $a > b > 0, \alpha > \beta > 0$ .

20. Compute the  $n$ -dimensional measure of the set:

$$\{(x_1, x_2, \dots, x_n) : x_j \geq 0, j = 1, \dots, n, x_1 + 2x_2 + 3x_3 + \cdots + nx_n \leq n\}$$

21. Important items since the last midterm:

- (a) Riemann integral and its basic properties
- (b) Jordan measure
- (c) Fubini's theorem
- (d) Change of variables formula
- (e) Arc length
- (f) Line integrals
- (g) Green's theorem