This document contains a proof of the equality of mixed partials under a natural assumption. The theorem is due to W. H. Young [1], but his proof is hard to follow. I hope this proof is easy to follow. It is essentially due to Dieudonné. There is a nice exposition in Pugh’s text. This exposition is that proof, with perhaps simpler notation.

**Theorem 1.** Suppose \( f(x, y) \) is defined in a neighborhood of a point \((a, b)\). Suppose the partial derivatives \( f_x, f_y \) are defined in a neighborhood of \((a, b)\) and are differentiable at \((a, b)\). (In particular this implies that \( f_x, f_y \) are continuous at \((a, b)\), but it is not assumed that their derivatives exist anywhere other than at \((a, b)\).) A short statement of the assumption is that \( Df = [f_x, f_y] \) is differentiable at \((a, b)\). This is sometimes stated as \( f \) is twice differentiable at \((a, b)\). Then

\[
(f_x)_y(a, b) = (f_y)_x(a, b),
\]

sometimes stated as

\[
f_{xy}(a, b) = f_{yx}(a, b).
\]

**Proof.** Consider the function

\[
\Delta(t) = \left[ f(a + t, b + t) - f(a + t, b) \right] - \left[ f(a, b + t) - f(a, b) \right].
\]

If we let \( g(s) = f(a + s, b + t) - f(a + s, b) \)

\[
\Delta(t) = g(t) - g(0).
\]

By the mean value theorem

\[
\Delta(t) = g(t) - g(0) = tg'(\xi) \quad (1)
\]

\[
= [f_x(a + \xi, b + t) - f_x(a + \xi, b)]t \quad (2)
\]

\[
= \left[ f_x(a, b) + (f_x)_x(a, b)\xi + (f_x)_y(a, b)t + p_1(t)\xi + p_2(t)t \right]t \quad (3)
\]

\[
- \left[ f_x(a, b) + (f_x)_x(a, b)\xi + q_1(t)\xi \right]t \quad (4)
\]

\[
= (f_x)_y(a, b)t^2 + [p_1(t)\xi + q_1(t)\xi + p_2(t)t]t. \quad (5)
\]

Now divide by \( t^2 \) and remember that \( |\xi| < |t| \) to get

\[
\frac{\Delta(t)}{t^2} = (f_x)_y(a, b) + \frac{[p_1(t)\xi + q_1(t)\xi + p_2(t)t]}{t}.
\]

The last term goes to 0 as \( t \to 0 \). Hence

\[
(f_x)_y(a, b) = \lim_{t \to 0} \frac{\Delta(t)}{t^2}.
\]

The argument is symmetric in \( x \) and \( y \), so

\[
(f_x)_y(a, b) = \lim_{t \to 0} \frac{\Delta(t)}{t^2} = (f_y)_x(a, b).
\]
Neat, huh!