lim sup and lim inf

October 16, 2015

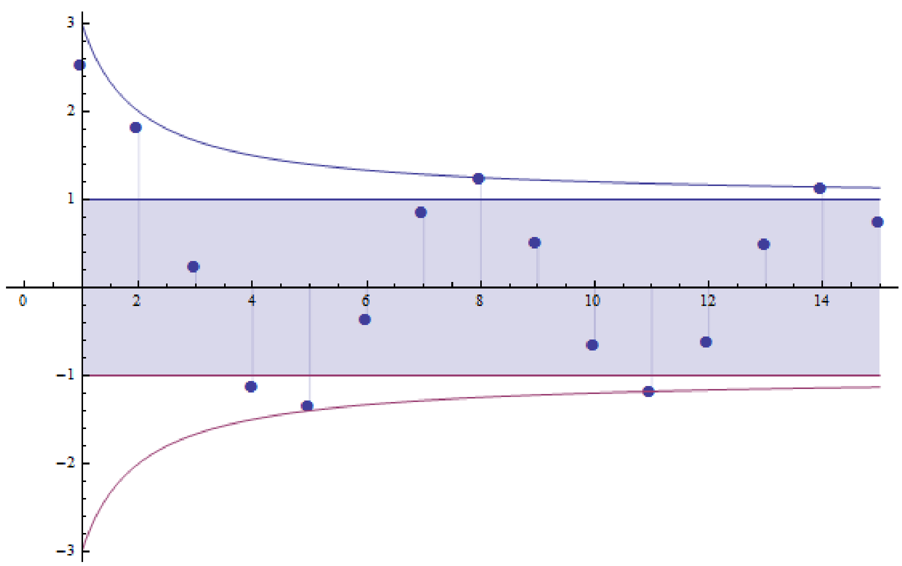
I didn't finish the proof of problem 1.5.9 in class today, but it is important to know about.

Once we've defined $Y_k = \sup\{x_k, x_{k+1}, x_{k+2}, ...\}$, then Y_k is a decreasing, bounded sequence, so it converges to the infimum of its values; call it a. Then the two properties of infimum (actually supremum, but just flip some signs and inequalities) at the top of page 25 correspond to the two properties you are asked to prove:

- We have that $Y_k \geq a$ for every k. For some $\epsilon > 0$, if there were only finitely many k with $x_k > a - \epsilon$, we could pick some K larger than any of them, and then $x_K, x_{K+1}, ...$ would all be less than or equal to $a - \epsilon$, as would Y_K, a contradiction.

- We have that for every $\epsilon > 0$, there is a Y_k with $Y_k < a + \epsilon$, which implies that x_k, x_{k+1}, etc. are all less than $a + \epsilon$, and there can only be finitely many which are greater.

I find thinking of the lim sup as the "inf of the sups" to be really unintuitive. Perhaps a better way of thinking about it this: if you plot the points of the sequence on a graph and the sequence has a limit, the points will get closer and closer to a horizontal line at the limit the further out you go. If the sequence doesn't have a limit, but is bounded, then there isn't a single line, but a strip which it comes closer and closer to being contained in.



Here is an example, with the sequence $(\sin k)(1 + 1/k)$. The sequence does not have a limit, but it does have a lim sup of 1 and a lim inf of -1, and you can see how the points get closer to the shaded strip as k increases.

Since lim sup and lim inf give asymptotic information without being as restrictive as the definition of limit, they are useful in various contexts, particularly in number theory. You can't easily use limits to talk about the asymptotics of gaps between prime numbers, since they jump all over the place, but you can talk about the lim sup and inf of functions related to them. lim sup also shows up in the most general forms of the ratio test and root test, which you'll be covering later this year.