Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover up to §4.1.

1. Is the set $\{(x,y) : y^2 + x^2 e^y = 0\}$ a smooth curve? Is the set $\{(a\cos t, b\sin t) : t \in (0,\pi)\}$, where $a > 0, b > 0$ a smooth curve?

2. Expand $(1 - x + 2y)^3$ in powers of $x - 1$ and $y - 2$ in two different ways. The first way is by using algebra and the second way is by computing the Taylor series.

3. Let f be defined and bounded on $[a,b]$. Define a function g on $[a,b]$ by the formula $g(x) = \overline{I}(\chi_{[a,x]} f)$. In other words $g(x)$ is the upper integral of f on the interval $[a,x]$. Prove that g is continuous on $[a,b]$. Suppose f is continuous at x_0. Prove that $g'(x_0) = f(x_0)$. The same is true for lower integrals.

4. Using the method of Lagrange multipliers, find the highest and lowest points of the circle
$$x^2 + y^2 + z^2 = 16,\ (x+1)^2 + (y+1)^2 + (z+1)^2 = 27$$

5. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.

6. Let $a > 0$ and $b > 0$. Decide whether or not the map $F(r,t) = (ra\cos t, rb\sin t)$ from $\{(r,t) : 0 < r < 1, 0 < t < \pi/2\}$ to $\{(x,y) : x > 0, y > 0, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1\}$ has a differentiable inverse.

7. Let f be continuously differentiable on $[a,b]$ and assume $f'(x) > 0$ on $[a,b]$.
   (a) Prove that f has a continously differentiable inverse g and that $g'(x) > 0$.
   (b) Prove that
   $$\int_a^b f + \int_{f(a)}^{f(b)} g = bf(b) - af(a).$$
   Can you give a geometric interpretation of the result?

8. Let $f(x,y) = \sec(x + y^2)$. Find the first two non-zero terms in the Taylor series of $\cos x$, centered at 0. Use it to find the first two non-zero terms of the Taylor series of $\sec x$ centered at 0. Then use that series to find the first two non-zero terms of f at $(0,0)$.

9. Let g be a polynomial of degree three. Prove that

$$\int_{-1}^{1} g = \frac{g(-1) + 4g(0) + g(1)}{3}.$$

10. Consider the following function

$$F(x, y) = (\frac{x}{1 + x + y}, \frac{y}{1 + x + y}),$$

which has the set $\{(x, y) : 1 + x + y \neq 0\}$ as its domain. Compute $\frac{\partial(f, g)}{\partial(x, y)}$. Where is it different from 0? Show that F is $1 - 1$ and find an explicit formula for its inverse. Use these results to describe the exact image of F

11. Folland, §2.9, problem 16.

12. Let f be a positive continuous function on $I = [a, b]$. Let $M = \max\{f(x) : x \in I\}$. Prove that

$$\lim_{n\to\infty} \left(\int_I f^n\right)^{1/n} = M.$$

13. Suppose $F(x, y)$ is a C^2 function that satisfies the equations $F(x, y) = F(y, x), F(x, x) = x$. Prove that the quadratic term in the Taylor polynomial of F based at the point (a, a) is $\frac{1}{2}F_{xx}(a, a)(x - y)^2$.

14. There may be homework problems or example problems from the text or lectures on the midterm.