The fundamental theorem of algebra was stated in various forms going back even before Euler. A variety of proofs were proposed. Gauss gave several proofs, not all of them correct. I have not read any of his proofs, nor have I read the proof of Jean-Robert Argand. It is claimed that Argand gave the first correct proof in 1814. I've not been able to read his exact proof, but from what I know of it the proof I am going to give here is pretty close to it. It is totally elementary except for using the fact that a continuous function on a compact set assumes a minimum. Argand must have implicitly assumed that, since he didn’t have a construction of the real numbers (for example the least upper bound axiom).

**Theorem 1 (Fundamental Theorem of Algebra).** Let \( f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \), where \( z \) is a complex variable and the \( a_j \) are complex numbers. Then there is a complex number \( a \in \mathbb{C} \) for which \( f(a) = 0 \).

**Proof.** Assume not. Then \( f(z) \neq 0 \) for all \( z \in \mathbb{C} \). Consider \( |f| \) on the set \( K = \{ z : |f(z)| \leq |f(0)| \} \). Since \( |f(z)| \to \infty \) as \( |z| \to \infty \), \( K \) is a compact set (closed and bounded). Since \( K \) is compact, \( |f| \) assumes a minimum on \( K \) and this will be a global minimum of \( |f| \) on all of \( \mathbb{C} \). If the minimum is assumed at \( a \), we can change variables and let \( p(z) = f(z + a) \) to get a new polynomial with a minimum at 0. We can multiply \( p \) by a constant to arrange it so the minimum value is 1. Let's call the new polynomial \( q \). Then here is how \( q \) looks

\[
q(z) = 1 + b_k z^k + \cdots + b_n z^n,
\]

where the minimum value of \( |q| \) is 1 and it is assumed at 0, where in this expression it is assumed that \( b_k \neq 0 \). This is all done to make the algebra transparent. Now we compare the terms after \( b_k z^k \) to \( b_k z^k \).

**Claim 1.** There is an \( \epsilon > 0 \) so that

\[
|b_k z^k| \geq 2(|b_{k+1} z^{k+1}| + \cdots + |b_n z^n|), \text{ when } |z| \leq \epsilon.
\]

**Proof of claim.** We can divide by \( |z| \) and the claim is equivalent to

\[
|b_k| \geq 2(|b_{k+1} z^k| + \cdots + |b_n z^{n-1}|), \text{ when } |z| \leq \epsilon,
\]

which is true since the right side goes to 0 as \( |z| \to 0 \).

Now choose \( z \) to be so that \( b_k z^k = -\delta \), where \( \delta \leq |b_k| \epsilon^k \). Then \( |z|^k \leq \frac{\delta}{|b_k|} \leq \epsilon^k \). Hence

\[
\delta \geq 2(|b_{k+1} z^k| + \cdots + |b_n z^{n-1}|).
\]

Finally choose \( \delta \) so that \( 0 < \delta < 1 \). Then for this \( z \)

\[
|q(z)| = |1 - \delta + b_{k+1} z^{k+1} + \cdots + b_n z^n| \leq 1 - \delta + \frac{\delta}{2}
\]

\[
= 1 - \frac{\delta}{2}
\]

\[
< 1
\]

\[
\rightarrow
\]