Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover up to §4.3.

1. Expand $(1 - x + 2y)^3$ in powers of $x - 1$ and $y - 2$ in two different ways. The first way is by using algebra and the second way is by computing the Taylor series.

2. Let $f$ be defined and bounded on $[a, b]$. Define a function $g$ on $[a, b]$ by the formula $g(x) = T(\chi_{[a,x]}f)$. In other words $g(x)$ is the upper integral of $f$ on the interval $[a, x]$. Prove that $g$ is continuous on $[a, b]$. Suppose $f$ is continuous at $x_0$. Prove that $g'(x_0) = f(x_0)$. The same is true for lower integrals.


4. Suppose $a < b < c < d$. Let $I = [a, b]$, $J = [c, d]$, $R = I \times J$ and let $f(x, y) = |x - y|$, if $x \in I$, $y \in J$. Compute $\int_R f$.

5. Using the method of Lagrange multipliers, find the highest and lowest points of the circle

   $x^2 + y^2 + z^2 = 16$, $(x + 1)^2 + (y + 1)^2 + (z + 1)^2 = 27$

6. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.
7. Find the shortest distance from the point \((1, -1, 1)\) to the surface \(z = xy\).

8. Let \(f\) be continuously differentiable on \([a, b]\) and assume \(f'(x) > 0\) on \([a, b]\).
   (a) Prove that \(f\) has a continuously differentiable inverse \(g\) and that \(g'(x) > 0\).
   (b) Prove that
   \[
   \int_a^b f + \int_{f(a)}^{f(b)} g = bf(b) - af(a).
   
   Can you give a geometric interpretation of the result?

9. Let \(f(x, y) = \sec(x + y^2)\). Find the first two non-zero terms in the Taylor series of \(\cos x\), centered at 0. Use it to find the first two non-zero terms of the Taylor series of \(\sec x\) centered at 0. Then use that series to find the first two non-zero terms of \(f\) at \((0, 0)\).

10. Let \(g\) be a polynomial of degree three. Prove that
    \[
    \int_{-1}^1 g = \frac{g(-1) + 4g(0) + g(1)}{3}.
    
11. Consider the following function
    \[
    F(x, y) = \left( \frac{x}{1 + x + y}, \frac{y}{1 + x + y} \right),
    
    which has the set \(\{(x, y) : 1 + x + y \neq 0\}\) as its domain. Compute \(\frac{\partial(f, g)}{\partial(x, y)}\). Where is it different from 0? Show that \(F\) is \(1 - 1\) and find an explicit formula for its inverse. Use these results to describe the exact image of \(F\).

12. Folland, §2.9, problem 16.
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13. Let \( f \) be a positive continuous function on \( I = [a, b] \). Let \( M = \max\{f(x) : x \in I\} \). Prove that

\[
\lim_{n \to \infty} \left( \int_I f^n \right)^{1/n} = M.
\]

14. Suppose \( F(x, y) \) is a \( C^2 \) function that satisfies the equations \( F(x, y) = F(y, x), F(x, x) = x \). Prove that the quadratic term in the Taylor polynomial of \( F \) based at the point \((a, a)\) is \( \frac{1}{2} F_{xx}(a, a)(x - y)^2 \).

15. There may be homework problems or example problems from the text or lectures on the midterm.