Fubini's Theorem

Note Title 11/17/2008

Let $R = [a,b] \times [c,d]$ and $f \in \mathcal{R}(R)$.

Let $f_y(x) = f(x,y)$. Then $\overline{I}f_y \in \mathcal{R}[c,d]$ and $\underline{I}f_y \in \mathcal{R}[c,d]$ and $\int_c^d \underline{I}f_y = \int_c^d \overline{I}f_y = \int_R f$.

Proof: Let P and Q be partitions of $[a,b]$ and $[c,d]$ respectively. They define a partition of R that I will denote by $P \times Q$ (bad notation). By definition

$\overline{I}f_y \le \sum_i M_i(y)|I_i|$, where $I_i = [x_{i-1}, x_i]$, $|I_i| = x_i - x_{i-1}$.

and $M_i(y) = \sup\{f_y(x) : x \in I_i\}$

$= \sup\{f(x,y) : x \in I_i\}$.

So

$$\sup\{\overline{I}f_y : y \in J_j\} \le \sup\{\sum_i M_i(y)|I_i| : y \in J_j\},$$

where $J_j = [y_{j-1}, y_j]$, $|J_j| = y_j - y_{j-1}$. (I should have noted that $P = \{x_0, \dots, x_n\}$, $Q = \{y_0, \dots, y_m\}$.)

Now $y \in J_j$, $M_i(y) \le M_{ij}$, where

$M_{ij} = \sup\{f(x,y) : x \in I_i, y \in J_j\}$. Next,

$$S_Q(\overline{I}f_y) = \sum_j \sup\{\overline{I}f_y : y \in J_j\} \cdot |J_j|$$
$$\le \sum_j \left(\sum_i M_{ij}|I_i|\right)|J_j| = S_{P\times Q}(f)$$

since $M_i(y) \le M_{ij}$. Hence

$$S_Q\left(\overline{I} f_y\right) \le S_{P\times Q}(f).$$

Similarly

$$s_{P\times Q}(f) \le s_Q\left(\underline{I} f_y\right).$$

Thus

$$s_{P\times Q}(f) \le s_Q\left(\underline{I} f_y\right) \le s_Q\left(\overline{I} f_y\right) \le S_Q\left(\overline{I} f_y\right) \le S_{P\times Q}(f),$$

$$s_{P\times Q}(f) \le s_Q\left(\underline{I} f_y\right) \le S_Q\left(\underline{I} f_y\right) \le S_Q\left(\overline{I} f_y\right) \le S_{P\times Q}(f).$$

Since we can choose P and Q so that

$S_{P\times Q}(f) - s_{P\times Q}(f) < \epsilon$, it follows that

$$S_Q\left(\overline{I} f_y\right) - s_Q\left(\overline{I} f_y\right) < \epsilon, \quad S_Q\left(\underline{I} f_y\right) - s_Q\left(\underline{I} f_y\right) < \epsilon.$$

Finally,

$$\int_R f = \int_c^d \overline{I} f_y = \int_c^d \underline{I} f_y.$$