CAUCHY-BINET

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Theorem 0.1. *(Cauchy-Binet) Let A be a $k \times n$ matrix and B be an $n \times k$ matrix. Then*

$$\det(AB) = \sum_J \det(A(J)) \det(B(J)),$$

where $J = (j_1, j_2, \ldots, j_k)$, $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, runs through all such multi-indices, $A(J)$ denotes the matrix formed from A using columns J (in that order), and $B(J)$ denotes the matrix formed using rows J of B in that order.

Proof. By definition of matrix product

$$\begin{aligned}\det AB &= \det \begin{bmatrix} \sum_{j_1=1}^n a_{1j_1} b_{j_1 1} & \ldots & \sum_{j_k=1}^n a_{1j_k} b_{j_k k} \\ \vdots & \vdots & \vdots \\ \sum_{j_1=1}^n a_{kj_1} b_{j_1 1} & \ldots & \sum_{j_k=1}^n a_{kj_k} b_{j_k k} \end{bmatrix} \\ &= \sum_{j_1,\ldots,j_k=1}^{n} \det(A(j_1, j_2, \ldots, j_k) b_{j_1 1} b_{j_2 2} \ldots b_{j_k k},\end{aligned}$$

by the multi-linearity of the determinant. Since $\det(A(j_1, j_2, \ldots, j_k) = 0$ if the indices j_i are not all distinct, only those sets of indices occur in the sum. For a fixed multi-index $J' = (j'_1, j'_2, \ldots, j'_k)$ with $1 \leq j'_1 < j'_2 < \cdots < j'_k \leq n$ and J some multi-index with these indices in some order, let $j'_i = j_{\sigma(i)}$ where σ is a permutation of $[n]$. Then

$$\det(A(j_1, j_2, \ldots, j_k) = \operatorname{sgn}(\sigma) \det(A(j'_1, j'_2, \ldots, j'_k).$$

Now let J' be fixed, and sum over all J which are permutations of J'. Let τ be the inverse of σ. Then $j_i = j_{\sigma\tau(i)} = j'_{\tau(i)}$. So the sum multiplying $\det(A(j'_1, j'_2, \ldots, j'_k) = \det(A(J'))$ is

$$\begin{aligned}&\sum_\sigma \operatorname{sgn}(\sigma) b_{j'_{\tau(1)} 1} b_{j'_{\tau(2)} 2} \ldots b_{j'_{\tau(k)} k} \\ &= \sum_\tau \operatorname{sgn}(\tau) b_{j'_{\tau(1)} 1} b_{j'_{\tau(2)} 2} \ldots b_{j'_{\tau(k)} k} \\ &= \det B(J').\end{aligned}$$

Hence

$$\det(AB) = \sum_{J'} \det(A(J')) \det(B(J')).$$

□

Corollary 0.1.

$$\det AA^T = \sum_J (\det A(J))^2.$$

Here's an application.

Corollary 0.2. *Let Π be a k-parallelepiped in $\mathbb{R}^n$ and let Π_J be the orthogonal projection of Π onto the k-dimensional subspace spanned by the x_J axes. Let $m_J = \mu(\Pi_J)$ be the k-dimensional measure of this k-parallelepiped. Then*

$$(\mu(\Pi)^2 = \sum_J m_J^2 = \sum_J \mu(\Pi_J)^2.$$

Proof. Recall that if $v_1, v_2, \ldots, v_k$ are the k row vectors which are the spanning edges of Π and V is the $k \times n$ matrix defined by

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix},$$

then the measure of Π is $\sqrt{\det VV^T}$. Now use the previous corollary and interpret each term in the sum as the square of the measure of a k-parallelepiped in $\mathbb{R}^k$. □

This is a sort of Pythagorean theorem, generalizing the length (1-dimensional measure) formula for a line segment in $\mathbb{R}^n$.

Application 0.1. *Let the surface S in $\mathbb{R}^4$ be defined by the parameterization $(x, y) \to (x, y, f(x, y), g(x, y)), (x, y) \in D \subset \mathbb{R}^2$. Then the area of S is*

$$\int_D \left(1 + f_x^2 + f_y^2 + g_x^2 + g_y^2 + \left(\frac{\partial(f, g)}{\partial(x, y))}\right)^2\right)^{1/2} dxdy.$$

The general result for this type of parameterization is as follows. Let $(x_1, x_2, \ldots, x_k) = x_K \in D \subset \mathbb{R}^k$ and let $(f_1(x_K), \ldots, f_m(x_K)) = f_M(x_K)$ be an m-tuple of differentiable functions defined on D. Let $\mathcal{M} = \{(x_K, f_M(x_K) : x_K \in D\}$. Then the k-dimensional measure of $\mathcal{M}$ is

$$\int_D \left(1 + \sum_{I,J,1 \le |I| = |J| \le k} \left(\frac{\partial f_I}{\partial x_J}\right)^2\right)^{1/2} dx_K.$$