Thomae's Function

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This note is a solution to problem 7 from §1.3. The function known as Thomae's function.

Theorem 1. *Let f be defined by*

$$f(x) = \begin{cases} \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ and } gcd(p,q) = 1 \text{ and } q > 0 \\ 0 \text{ if } x \text{ is irrational.} \end{cases}$$

Then f is discontinuous at the rationals and continuous at the irrationals.

Proof. Let r be irrational. Then $f(r) = 0$. Let m be a positive integer. Then r is in a unique interval of the form $(\frac{k}{m}, \frac{k+1}{m})$. Let $d_m = \min\{|r - \frac{k}{m}|, |r - \frac{k+1}{m}|\}$ and let $\delta_m = \min\{d_1, d_2, \ldots, d_m\}$. Notice $\delta_m < \frac{1}{m}$. Let $\epsilon > 0$ be given. Choose m so that $\frac{1}{m} < \epsilon$. Let $\delta = \delta_m$. If x is a rational number with $|x - r| < \delta$ then $x = \frac{p}{q}$ with $gcd(p,q) = 1$ and $q > m$. Hence $0 < f(x) = \frac{1}{q} < \frac{1}{m} < \epsilon$. If x is irrational, $f(x) = 0$. So for any x, with $|x - r| < \delta$, $|f(x) - f(r)| < \epsilon$. This proves that f is continuous at any irrational number.

Next let $r = \frac{p}{q}$ be rational. Then $f(r) = \frac{1}{q}$. The number $x_k = r + \frac{1}{k\sqrt{2}}$ is irrational, $|x_k - r| = \frac{1}{k\sqrt{2}}$ and $f(x_k) = 0$. Let $\epsilon = \frac{1}{2q}$. Is there a δ so that $|x - r| < \delta$ implies that $|f(x) - f(r)| = |f(x) - \frac{1}{q}| < \frac{1}{2q}$? No matter how small δ is there is an irrational number $x_k = r + \frac{1}{k\sqrt{2}}$ $|r - x_k| < \delta$, with $f(x_k) = 0$ and hence $|f_k(x) - f(r)| = \frac{1}{q} > \frac{1}{2q}$. □