Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems — I encourage you to do that. The test will cover up to §4.4.

1. Let $f$ be defined and bounded on $[a, b]$. Define a function $g$ on $[a, b]$ by the formula $g(x) = \overline{\int}(\chi_{[a,x]}f)$. In other words $g(x)$ is the upper integral of $f$ on the interval $[a, x]$. Prove that $g$ is continuous on $[a, b]$. Suppose $f$ is continuous at $x_0$. Prove that $g'(x_0) = f(x_0)$. The same is true for lower integrals.

2. Folland, §4.1, # 9.

3. Suppose $a < b < c < d$. Let $I = [a, b]$, $J = [c, d]$, $R = I \times J$ and let $f(x, y) = |x - y|$, if $x \in I$, $y \in J$. Compute $\int_R f$.

4. Using the method of Lagrange multipliers, find the highest and lowest points of the circle

$$x^2 + y^2 + z^2 = 16, \quad (x + 1)^2 + (y + 1)^2 + (z + 1)^2 = 27$$

5. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.
6. Let \( f(x, y) = \sec(x + y^2) \). Find the first two non-zero terms in the Taylor series of \( \cos x \), centered at 0. Use it to find the first two non-zero terms of the Taylor series of \( \sec x \) centered at 0. Then use that series to find the first two non-zero terms of \( f \) at \((0, 0)\).

7. Let \( g \) be a polynomial of degree three. Prove that
\[
\int_{-1}^{1} g = \frac{g(-1) + 4g(0) + g(1)}{3}.
\]

8. Consider the following function
\[
F(x, y) = \left( \frac{x}{1 + x + y}, \frac{y}{1 + x + y} \right),
\]
which has the set \( \{(x, y) : 1 + x + y \neq 0\} \) as its domain. Compute \( \frac{\partial(f, g)}{\partial(x, y)} \). Where is it different from 0? Show that \( F \) is \( 1 - 1 \) and find an explicit formula for its inverse. Use these results to describe the exact image of \( F \).


10. Let \( f \) be a positive continuous function on \( I = [a, b] \). Let \( M = \max\{f(x) : x \in I\} \). Prove that
\[
\lim_{n \to \infty} \left( \int_{I} f^{n} \right)^{1/n} = M.
\]

11. Suppose \( F(x, y) \) is a \( C^2 \) function that satisfies the equations \( F(x, y) = F(y, x), F(x, x) = x \). Prove that the quadratic term in the Taylor polynomial of \( F \) based at the point \((a, a)\) is \( \frac{1}{2}F_{xx}(a, a)(x - y)^2 \).

12. There may be homework problems or example problems from the text or lectures on the midterm.