The Divergence Theorem

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This note will explain the following theorem.

Theorem 1. *Let Ω be a relatively compact open set in $\mathbf{R}^n$, and assume $\partial\Omega$ is smooth. Let $\mathbf{F}$ be a smooth vector field defined on a neighborhood of $\overline{\Omega}$ and let $\mathbf{n}$ be the outward pointing unit normal to $\partial\Omega$. Let $d\mu$ be $(n-1)$-dimensional measure on $\partial\Omega$. Then*

$$\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\; d\mu = \int \text{div } \mathbf{F}. \tag{1}$$

The following is a discussion of this formula. We will use the special case of the Binet-Cauchy formula (page 8-9 of [1]

Theorem 2. *Let A be an $n \times (n-1)$ matrix. Let A_i be the submatrix of A with row i deleted. Then*

$$\det A^T A = \sum_{i=1}^{n} (\det A_i)^2. \tag{2}$$

Now let $\mathbf{X}(u_1, u_2, \ldots, u_{n-1})$ be a parametrization of a surface S in $\mathbf{R}^n$. If we let $A = [\mathbf{X}_{u_1}, \mathbf{X}_{u_2}, \ldots, \mathbf{X}_{u_{n-1}}]$, then this formula becomes

$$\det A^T A = \sum_{i=1}^{n} \left(\frac{\partial(x_1,\ldots \widehat{x_i},\ldots,x_n)}{\partial(u_1,\ldots,u_n)}\right)^2. \tag{3}$$

The symbol $\widehat{x}$ means omit x. It is convenient to introduce the notation

$$j_i = (-1)^{i-1}\frac{\partial(x_1,\ldots \widehat{x_i},\ldots,x_n)}{\partial(u_1,\ldots,u_n)}. \tag{4}$$

Then equation (3) reads

$$\det A^T A = \sum_{i=1}^{n} j_i^2. \tag{5}$$

It's also convenient to introduce

$$\mathbf{N} = (j_1, j_2, \ldots, j_n). \tag{6}$$

The vector $\mathbf{N}$ is orthogonal to S. Using this notation the formula for surface measure becomes

$$\mu(S) = \int_R \left(\sum_{i=1}^{n} j_i^2\right)^{1/2} = \int_R |\mathbf{N}| du = \int_R d\mu. \tag{7}$$

where R is the domain of the parameters. Suppose $S = \partial\Omega$ and that the parametrization is chosen so that $\mathbf{N}$ points out of S. Then $\mathbf{n} = \dfrac{\mathbf{N}}{|\mathbf{N}|}$ is the outward pointing unit normal. The divergence theorem becomes

Theorem 3.

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} d\mu = \int_{R} \mathbf{F} \cdot \mathbf{N} du = \int_{\Omega} \text{div } \mathbf{F}. \tag{8}$$

Let's look at the case $n = 2$. $\mathbf{X}(t) = (x(t), y(t))$ and $\mathbf{N} = (y'(t), -x'(t)$ (t is the parameter). Let $\mathbf{F} = (P, Q)$. Then equation (8) says

$$\int (P\frac{dy}{dt} - Q\frac{dx}{dt})dt = \int_{\partial\Omega} Pdy - Qdx = \int_{\Omega} P_x + Q_y. \tag{9}$$

Next let $(x(u,v), y(u,v), z(u.v))$ be a parametrization of a surface and let $\mathbf{F} = (P, Q, R)$. Then

$$\int \left(P\frac{\partial(y,z)}{\partial(u,v)} - Q\frac{\partial(x,z)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)} \right) dudv = \int_{\partial\Omega} Pdydz - Qdxdz + Rdxdy = \int_{\Omega} P_x + Q_y + R_z. \tag{10}$$

For these equations to be correct, parameterizations must be positive and appropriate interpretations must be given to $dydz, dxdz, dxdy$. Order is important.

References

[1] F. R. Gantmacher, *The Theory of Matrices, Volume I*, Chelsea, 1977.