Areas of Hypersurfaces

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This note will derive the following result.

Theorem 1. *Let $f(x_1, \ldots, x_n)$ be a differentiable function. Let $S = \{x_{n+1} = f(x_1, \ldots, x_n)\} \subset \mathbf{R}^{n+1}$. Then the n-dimensional area of S is*

$$A(S) = \int (1 + |\nabla f|^2)^{1/2}. \tag{1}$$

This theorem will rely on the following formula.

Theorem 2. *Let Π be the m-dimensional parallelotope in $\mathbf{R}^n$ with coterminous edges given by the vectors $v_1, \ldots, v_m$. Then the m-dimensional area of Π is*

$$A(\Pi) = (\det(C))^{1/2}, \tag{2}$$

where C is the $m \times m$ matrix with entries $v_i \cdot v_j$.

Proof. (of Theorem 2) Notice that we can write C in the following form.

$$C = V^T V, \ V = [v_1, \cdots, v_m] \tag{3}$$

where V is the $n \times m$ matrix matrix whose columns are the column vectors $v_1, \ldots, v_m$.. If Π is rectangular (the v_i are orthogonal), the matrix C is diagonal and the diagonal entries are $|v_i|^2$; so the result is true in this case. C is symmetric and it is easy to see it is positive semi-definite, since

$$\sum v_i \cdot v_j x_i x_j = \left| (\sum_1^m x_i v_i) \right|^2.$$

Hence $\det C \geq 0$. (If the vectors are linearly dependent (Π is degenerate), then $\sum_1^m x_i v_i = 0$ for some choice of the x_i and $\det C = 0$.) So assume the v_i are linearly independent. Let's modify v_1 by subtracting multiples of $v_2, \ldots, v_m$ to make it orthogonal to each of $v_2, \ldots, v_m$ This requires solving the equations

$$\sum_2^m a_j v_j \cdot v_k = v_1 \cdot v_k, \ k = 2, \ldots, m.$$

Since $v_2, \ldots, v_m$ are linearly independent, this system has a unique solution. If we let $\overline{v}_1 = \sum_2^m a_j v_j$ we get a new parallelotope with the same area. The new matrix $\overline{C}$ is related to the original matrix C by the formula

$$\overline{C} = A^T C A, \tag{4}$$

where A is the matrix that differs from the identity matrix by having the last $n-1$ entries in the first column replaced by $a_2, a_3, \dots, a_n$. The determinant of A is 1, so $\det(\overline{C}) = \det(C)$. Eventually this process produces a rectangular parallelotope $\overline{\Pi}$ with the same area as Π and thus

$$A(\overline{\Pi}) = A(\Pi) = (\det(\overline{C}))^{1/2} = (\det(C))^{1/2} \tag{5}$$

□

Theorem 1 is a consequence of the following lemma. To simplify notation we use vertical bars to denote determinant, $|A| = \det(A)$.

Lemma 1. *In the statement of the lemma, d denotes the determinant.*

$$1 + a_1^2 + a_3^2 + \cdots + a_n^2 = \begin{vmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \dots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \dots & a_2a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \dots & 1+a_n^2 \end{vmatrix} = d \tag{6}$$

Proof. The proof is by induction on n.

$$d = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \dots & a_2a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \dots & 1+a_n^2 \end{vmatrix} + a_1 \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \dots & a_2a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \dots & 1+a_n^2 \end{vmatrix} \tag{7}$$

Now by subtracting appropriate multiples of the first row from the other rows in the second determinant we get

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \dots & a_2a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \dots & 1+a_n^2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = a_1 \tag{8}$$

Hence $d = (1 + a_2^2 + \dots a_n^2) + a_1^2$. □

Proof. (of Theorem 1). According to the definition of area, the area of the surface is

$$A(S) = \int A(\Pi) dx_1 dx_2 \dots dx_n \tag{9}$$

Where Π is the parallelotope spanned by the column vectors $[1, 0, 0, \dots, f_{x_1}]^T, [0, 1, 0, \dots, f_{x_2}]^T, \dots, [0, 0, \dots, 1, f_{x_n}]^T$. Theorem 2 and Lemma 1 tell how to compute $A(\Pi)$ and the result is $A(\Pi) = (1 + f_{x_1}^2 + f_{x_2}^2 + \dots f_{x_n}^2)^{1/2} = (1 + |\nabla f|^2)^{1/2}$. □