Solving for the Limit of a Recursively Defined Sequence

Owen Biesel

October 5, 2006

Let \( x_k \) be defined recursively by \( x_1 = \frac{1}{2} \),

\[
x_{k+1} = \begin{cases} 
\frac{1}{2} + \frac{x_k}{2}, & \text{if } x_k < 1; \\
\frac{1}{2}, & \text{if } x_k \geq 1.
\end{cases}
\]

Then \( x_k \to 1 \) as \( k \to \infty \), but this solution cannot be obtained by letting \( x_{k+1} = L = x_k \) in the recursion relation.

**Proof.** We will show that \( x_k = 1 - 2^{-k} \) by induction on \( k \). The base case is easily verified: \( x_1 = \frac{1}{2} = 1 - \frac{1}{2} \). Now suppose \( x_k = 1 - 2^{-k} \) for some \( k \). Then \( x_k < 1 \), so

\[
x_{k+1} = \frac{1}{2} + \frac{x_k}{2} = \frac{1}{2} + \frac{1 - 2^{-k}}{2} = \frac{1}{2} + \frac{1}{2} - 2^{-k-1} = 1 - 2^{-(k+1)}
\]

Therefore \( x_k = 1 - 2^{-k} \) for all \( k \). Hence \( \lim_{k \to \infty} x_k = 1 - 0 = 1 \).

Now let \( x_{k+1} = L = x_k \) in the recursion relation, so that

\[
L = \begin{cases} 
\frac{1}{2} + \frac{L}{2}, & \text{if } L < 1; \\
\frac{1}{2}, & \text{if } L \geq 1.
\end{cases}
\]

If \( L < 1 \), then \( L = \frac{1}{2} + \frac{L}{2} > \frac{L}{2} + \frac{L}{2} = L \), a contradiction. Thus \( L \geq 1 \), so the recursion relation tells us that \( L = 2 \). Since \( L \neq \lim_{k \to \infty} x_k \), we therefore cannot always use a recursion relation directly as a condition on the limit of the recursively defined sequence. \( \Box \)