Solving for the Limit of a Recursively Defined Sequence

Owen Biesel

October 5, 2006

Let x_k be defined recursively by $x_1 = 1/2$,

$$x_{k+1} = \begin{cases} \frac{1}{2} + \frac{x_k}{2}, & \text{if } x_k < 1; \\ 2, & \text{if } x_k \geq 1. \end{cases}$$

Then $x_k \to 1$ as $k \to \infty$, but this solution cannot be obtained by letting $x_{k+1} = L = x_k$ in the recursion relation.

Proof. We will show that $x_k = 1 - 2^{-k}$ by induction on k. The base case is easily verified: $x_1 = \frac{1}{2} = 1 - \frac{1}{2}$. Now suppose $x_k = 1 - 2^{-k}$ for some k. Then $x_k < 1$, so

$$\begin{aligned} x_{k+1} &= \frac{1}{2} + \frac{x_k}{2} \\ &= \frac{1}{2} + \frac{1 - 2^{-k}}{2} \\ &= \frac{1}{2} + \frac{1}{2} - 2^{-k-1} \\ &= 1 - 2^{-(k+1)} \end{aligned}$$

Therefore $x_k = 1 - 2^{-k}$ for all k. Hence $\lim_{k\to\infty} x_k = 1 - 0 = 1$.

Now let $x_{k+1} = L = x_k$ in the recursion relation, so that

$$L = \begin{cases} \frac{1}{2} + \frac{L}{2}, & \text{if } L < 1; \\ 2, & \text{if } L \geq 1. \end{cases}$$

If $L < 1$, then $L = \frac{1}{2} + \frac{L}{2} > \frac{L}{2} + \frac{L}{2} = L$, a contradiction. Thus $L \geq 1$, so the recursion relation tells us that $L = 2$. Since $L \neq \lim_{k\to\infty} x_k$, we therefore cannot always use a recursion relation directly as a condition on the limit of the recursively defined sequence. $\square$