1. (10 pts)

(a) Find the solution to \( \frac{dy}{dt} = te^t(y^2 - 1) \) with \( y(0) = 2 \).

\[
\int \frac{dy}{y^2 - 1} = \int te^t \, dt \\
\text{LHS: } \frac{1}{y^2 - 1} = \frac{A}{y - 1} + \frac{B}{y + 1} \\
\int \frac{1}{y^2 - 1} \, dy = \frac{1}{2} \int \frac{dy}{y - 1} - \frac{1}{2} \int \frac{dy}{y + 1} \\
= \frac{1}{2} \ln |y - 1| - \frac{1}{2} \ln |y + 1| = \frac{1}{2} \ln \left| \frac{y - 1}{y + 1} \right| \\
\text{RHS: } \int te^t \, dt = te^t - \int e^t \, dt = te^t - e^t + C \\
\frac{1}{2} \ln \left| \frac{y - 1}{y + 1} \right| = te^t - e^t + C \\
y(0) = 2 \Rightarrow \frac{1}{2} \ln \left( \frac{1}{3} \right) = -1 + C \\
c = \frac{1}{2} \ln \left( \frac{1}{3} \right) + 1
\]

(b) Find the general solution to \( \frac{dy}{dx} = \cos x - \frac{1}{x}y \). Assume \( x > 0 \).

\[
\frac{dy}{dx} + \frac{1}{x} y = \cos x \\
\text{I.F: } I(x) = e^\int \frac{1}{x} \, dx = e^{\ln x} = x \\
x \frac{dy}{dx} + y = x \cos x \Rightarrow \frac{d}{dx} (xy) = x \cos x \\
\Rightarrow xy = \int x \cos x \, dx = x \sin x - \int \sin x \, dx \\
= x \sin x + \cos x + C \\
\Rightarrow y = \frac{\sin x + \cos x + C}{x}
\]
2. (5 pts)
Find the value of \( b \) for which the given equation is exact, and then solve it using that value of \( b \).

\[
(xy^2 + bx^2y) + (x + y)x^2 \frac{dy}{dx} = 0
\]

\[
\frac{\partial M}{\partial y} = 2xy + bx^2,
\]
\[
\frac{\partial N}{\partial x} = 3x^2 + 2xy.
\]

To be exact:

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies bx^2 + 2xy = 3x^2 + 2xy \implies b = 3
\]

Differential equation:

\[
(xy^2 + 3x^2y) + (x^3 + 3y^2x^2) \frac{dy}{dx} = 0
\]

\[
\int M \, dx = \int (xy^2 + 3x^2y) \, dx = \frac{x^2y^2}{2} + x^3y + C_1(x)
\]

\[
\int N \, dy = \int (x^3 + 3y^2x^2) \, dy = x^3y + \frac{y^2}{2}x^4 + C_2(y)
\]

\[
G(x,y) = C
\]

\[
\frac{1}{2} x^2y^2 + x^3y = C
\]

is the general solution.
3. (15 pts)

(a) A baseball is dropped from an airplane. The mass of the baseball is about 0.2 kg. The force due to air resistance is proportional, and in opposite direction, to the velocity with proportionality constant $k$ (where $k > 0$).

Just like what we did in homework assume that there are two forces acting on the ball: the force due to gravity and the force due to air resistance. (Recall: Newton’s second law is $ma = F$ and the acceleration due to gravity is 9.8 meters/second$^2$).

i. Give the differential equation and the initial condition for the velocity $v(t)$. (Do not solve)

\[
F_{net} = ma = mg - kv = -9.8 - ku
\]

\[
\frac{dv}{dt} = -9.8 - \frac{k}{0.2}v
\]

\[
v(0) = 0
\]

ii. The limit of $v(t)$ as $t \to \infty$ is called the terminal velocity. For a baseball the terminal velocity is known to be about 42 meters/second. Using this fact, find the value of $k$.

\[
-9.8 - \frac{k}{0.2}(42) = 0
\]

\[
\Rightarrow \quad \frac{42}{0.2} \quad k = 9.8
\]

\[
k = 0.0467
\]

(b) For this part only assume that $k = 0.05$. Use Euler’s method with step size $h = 0.25$, estimate $v$ after 1 second.

\[
\begin{array}{c|c|c}
\text{t} & v & f(t, v) \\
\hline
0 & 0 & -9.8 \\
0.25 & -2.45 & -9.1875 \\
0.5 & -4.75 & -8.6125 \\
0.75 & -6.8876 & -8.0751 \\
1 & -8.9147 & \approx v(1)
\end{array}
\]
4. (10 pts)

(a) Consider \( \frac{dy}{dt} = (y - 2)^{1/3} \) with \( y(0) = 2 \).

The solution is NOT guaranteed to be unique (you should know why).
Give two different solutions to this differential equation.

\[
\int \frac{dy}{(y - 2)^{1/3}} = \int dt \quad \Rightarrow \quad \frac{3}{2} (y - 2)^{2/3} = t + c \quad \Rightarrow \quad c = 0
\]
\[
\Rightarrow (y - 2)^{2/3} = \frac{2}{3} + t
\]
\[
\Rightarrow y - 2 = \left( \frac{2}{3} + t \right)^{3/2}
\]
\[
\Rightarrow y = 2 + \left( \frac{2}{3} + t \right)^{3/2}
\]

(b) Consider \( \frac{dy}{dt} = (y - 2)^2 \) with \( y(0) = 2 \).

Explain why the solution IS guaranteed to be unique.
Give the unique solution.

\( f(y) = (y - 2)^2 \) continuous everywhere
\[
\frac{df}{dy} = 2(y - 2)
\]

By Existence & Uniqueness Theorem, the solution is guaranteed to be unique.

\( y = 2 \) is the unique solution.
5. (10 pts) A pump attached to a reservoir pumps water into the reservoir at a rate of 2000$t$ liters per hour, where $t$ is measured in hours. At time $t = 0$ a valve at the bottom of the reservoir is opened and it begins to drain at a rate proportional to the amount of water in the reservoir. The reservoir initially contains 10000 liters of water, and the initial outflow rate is measured to be 1000 liters per hour.

(a) Establish an initial value problem that models the volume of water in the reservoir at time $t$.

\[
\frac{dV}{dt} = \text{Rate IN} - \text{Rate Out} - \\
\text{Rate IN} = 2000t \\
\text{Rate Out} = KV \\
V(0) = 10000 \\
K = \frac{1000}{10000} = 0.1 \\
\Rightarrow KV = 0.1V
\]

(b) Solve the initial value problem to find the number of liters of water in the reservoir at time $t$.

\[
\frac{dV}{dt} + 0.1V = 2000t \\
I. F. M: \mu(t) = e^{\int 0.1 dt} = e^{0.1t} \\
e^{0.1t} \frac{dV}{dt} + 0.1e^{0.1t} V = e^{0.1t} 2000t \\
\frac{d}{dt} (e^{0.1t} V) = 2000 e^{0.1t} \\
e^{0.1t} V = 2000 \int e^{0.1t} dt = 2000 \left( \frac{e^{0.1t}}{0.1} - \int_0^t e^{0.1s} ds \right) \\
e^{0.1t} V = 20000 + 200000 e^{0.1t} - 200000 e^{0.1t} \\
V = 20000t - 20000 + 10000 e^{-0.1t}
\]