The Poincaré Conjecture and the h-cobordism theorem

The generalized Poincaré Conjecture is now a theorem:

Theorem 0.1 *Let M be a smooth compact n-manifold homotopy-equivalent to S^n. Then M is homeomorphic to S^n.*

M need not be diffeomorphic to S^n, however, as discussed in an earlier lecture. The original conjecture of Poincaré was the case $n = 3$, formulated as follows:

Every simply-connected compact 3-manifold is homeomorphic to S^3.

It turns out that every 3-manifold admits the structure of smooth manifold, but we won't worry about this point; we are considering only smooth manifolds. To connect the original version with the above theorem, one needs to show:

Lemma 0.2 *If M is a simply-connected compact 3-manifold, then M is homotopy-equivalent to S^3.*

For those who are familiar with basic homotopy theory, I'll sketch the proof (those who are not can ignore it). Since $\pi_1 M = 0$, also $H_1 M = 0$. By Poincaré duality $H^2 M = 0$, and then by "universal coefficients" $H_2 M = 0$. Since $H_3 M \cong \mathbb{Z}$, the Hurewicz theorem yields a map $f : S^3 \longrightarrow M$ inducing an isomorphism on homology groups (this step needs M simply-connected, not just $H_1 M = 0$). Then the homology version of Whitehead's theorem shows that f is a homotopy-equivalence. This last step again requires simple-connectivity, and also that M has the homotopy-type of a CW-complex.

Remark: Poincaré's first version of the conjecture only assumed that M has the homology of S^3. But he himself soon found a counterexample, with fundamental group the binary isosahedral group of order 120. Recall here that H_1 is the abelianization of π_1, so if π_1 has trivial abelianization then homology doesn't see it.

Let's consider various cases of the theorem, in historical order.

$n = 1$. Every compact 1-manifold is homeomorphic to the circle. Although this seems intuitively "obvious", it's surprisingly tricky to prove rigorously. Try it if you don't believe me! If you assume the manifold is smooth, it's still a highly non-trivial exercise to get a diffeomorphism to S^1. See Milnor's *Topology from the differentiable viewpoint*, or Lee's text, Problem 15.13.

$n = 2$. Here the theorem is a special case of the classification theorem for compact surfaces. It's worth pointing out, however, that the proof of the surface theorem is extremely complicated; the hardest step, that of showing that surfaces can be triangulated, is hardly ever included in textbooks. For smooth surfaces there is a proof based on Morse theory; see Hirsch, *Differential Topology*. In any case, the difficulty of $n = 2$ is an ominous sign.

$n \geq 5$: This case was proved by Stephen Smale in 1961, using Morse theory. As far as I know, Smale's amazing proof has never been significantly simplified. It will take many lectures just to outline the argument, but it's worth it. One key point to be addressed, of course, is why Smale's argument breaks down in dimensions < 5.

$n = 4$: Proved by Michael Freedman in 1982. Completely different techniques are required; this case is beyond our scope.

$n = 3$: Proved by Grigori Perelman in 2002. Here the techniques are completely different from those of Smale or Freedman, with differential geometry playing a critical role. Again this case is beyond our scope.

1 Smale's h-cobordism theorem

The starting strategy of Smale's proof is easy to explain, and probably any of us could have gotten this far: Reeb's theorem says that if M admits a Morse function with only two critical points, then M is homeomorphic to a sphere. Moreover we know that any M admits *some* Morse function f. So what we need to show is that if M is homotopy-equivalent to S^n, then f can be modified so as to eliminate all but two critical points.

But there's a better way to reformulate this approach, replacing Reeb's theorem by the Regular Interval Theorem and in the process obtaining a much more general result. By removing two small disjoint discs we get a manifold W whose boundary is the disjoint union of two S^{n-1}'s. If M is homotopy-equivalent to S^n, then using "basic homotopy theory" as alluded to earlier, one can show that each of these boundary components is a deformation retract of W. This suggests that W is diffeomorphic to a cylinder $S^{n-1} \times I$. If true the Poincarè conjecture would follow immediately.

So let's consider the general question: Let W, V_0, V_1 be a *triad* as defined at the beginning of the course; i.e. W is a smooth compact manifold with boundary the disjoint union of the nonempty closed submanifolds V_0, V_1. In other words, each V_i is a union of boundary components of W; usually we are thinking of all three manifolds as connected, but this need not be the case in general. We call W an h-cobordism if V_0 and V_1 are deformation retracts of W, in which case we say that V_0 and V_1 are h-cobordant. The only example of such a thing that comes to mind—even after lengthy contemplation—is a cylinder $V_0 \times I$, or something diffeomorphic to such a cylinder (so V_0 would have to be diffeomorphic to V_1). This raises the:

h-cobordism question: Is every h-cobordism diffeomorphic to a cylinder? In particular, are h-cobordant manifolds diffeomorphic?

Examples due to Milnor show the answer is no in general. But Milnor's examples have non-trivial fundamental group, leaving open the possibility that the answer is yes in the simply-connected case. Here is Smale's 1961 theorem:

Theorem 1.1 *Let W be a simply-connected h-cobordism of dimension ≥ 6. Then W is diffeomorphic to a cylinder. In particular, h-cobordant manifolds of dimension ≥ 5 are diffeomorphic.*

If $\dim W = 5$ and the boundary components are ordinary spheres, the result still holds.

The Poincaré conjecture for $n \geq 5$ follows. However, examples due to Donaldson show that in general the simply-connected h-cobordism theorem fails for $n = 5$. For $n = 4$ it is still an open problem, equivalent to the "smooth Poincaré conjecture" in dimension 4. For $n = 3$ it follows from the 3-dimensional Poincaré conjecture proved by Perelman. For $n = 2$ the boundary components are necessarily circles, so we are no longer in the simply-connected case, but the we still get the h-cobordism theorem as a consequence of the classification of compact surfaces with boundary.

Reference: For the many details that will be omitted in the lectures, see Milnor's *Lectures on the h-cobordism theorem*, henceforth referred to as LHCT.

2 Morse functions on triads

Before opening the discussion of the proof of the h-cobordism theorem, we need to say a word about Morse functions on manifolds with boundary. We can define Morse functions exactly as we did in the boundaryless case, but without some restriction on the behaviour near the boundary they are not very useful. Given a triad (W, V_0, V_1), an *admissible* Morse function on it is a smooth $f : W \longrightarrow [a_0, a_1]$ such that (i) a_0, a_1 are regular values; (ii) $f^{-1}a_i = V_i$; and (iii) all critical points are nondegenerate. If you're worried about condition (i) at a boundary point, remember that we can always glue on collars to the boundary components, thereby obtaining a noncompact boundaryless manifold $\hat{W}$. Then f can be extended to $\hat{W}$ so that W is just $f^{-1}[a, b]$ as usual.

Lemma 2.1 *For any triad as above there exists an admissible Morse function f on W.*

For a proof see LHCT. Now, if there are no critical points then by the regular interval theorem we conclude that W is diffeomorphic to $V_0 \times I$. So the game now is to start with an h-cobordism, choose an admissible Morse function f on it, and then try to modify f so as to eliminate all critical points. With the hypotheses of Smale's theorem, this miraculously works, but only after many difficult steps. In our outline of the proof we will keep track carefully of when the three key hypotheses (1) W is an h-cobordism, (2) W is simply-connected, and (3) $n \geq 6$ are being used. In particular *we do not assume any of these three hypotheses except where explicitly stated.* To avoid silly counterexamples, however, we will assume $n \geq 3$.

3 Outline of the proof

The goal of the remaining lectures is to sketch the proof of Smale's theorem. The proof is long and intricate, but worth studying even if the Poincaré conjecture itself lies far from your main interests. For example, you will find here homology groups that live and breathe; especially striking is the realization of the cellular chain complex in terms of flows and intersection numbers (the latter being a geometric manifestation of Poincaré duality). To learn homology (or anything else in mathematics), you have to internalize it at a deep intuitive level. That means seeing the theory "in action", in interesting applications, and there are few applications more fascinating than this beautiful theorem of Smale.

I. Proof of the h-cobordism theorem: Preliminaries.

- *Gradient-like vector fields.* Given a Morse function f, a vector field X is gradient-like if (i) $Xf > 0$ away from critical points; and (ii) near critical points X looks like the gradient of f in a Morse chart. All references to "the flow" in the sequel refer to the flow associated to a gradient-like vector field.

- *Level-adjusting lemma.* Given a Morse function f and a critical point p, we can perturb f slightly so as to change the level of p without affecting its index or any of the other critical points.

- *Flow-adjusting lemma.* If $[a, b]$ is a regular interval for f, we know that the flow of any gradient-like vector field X yields a diffeomorphism $f^{-1}a \longrightarrow f^{-1}b$. This lemma will allow us to alter X so as to replace the given diffeomorphism by any desired diffeomorphism isotopic to it.

- *Isotopies and Thom's transversality theorem.* The flow-adjusting lemma will be used in the following way: We will want to arrange that under the flow in the previous bullet, certain embedded spheres in $f^{-1}a$ intersect transversally certain spheres in $f^{-1}b$. A famous theorem of Thom says that for any pair of submanifolds A, B in a manifold M, there is an isotopy h_t of the identity of M such that $h_1(A)$ is transverse to B. By flow-adjusting, any such isotopy of $f^{-1}b$ can be realized by the flow, and this will yield the transversality we want.

II. Rearrangement Theorem. This theorem is valid for any triad. Call a Morse function *self-indexing* if for every critical point p, $f(p) = \mathit{index}\, p$. The point of this condition is that the critical points appear in the right order: In other words, critical points of the same index appear at the same level, and $\mathit{index}\, p < \mathit{index}\, q$ if and only if $f(p) < f(q)$. Thus the cells in the associated cell decomposition are being attached in skeletal progression. We will show that given any Morse function f, it can be rearranged to be self-indexing, without changing the critical points or their indices. (The level-adjusting lemma does not suffice for this purpose.) From here on we assume our Morse function is self-indexing.

III. The Morse complex. The beautiful fact is that the boundary map in the cellular chain complex can now be described in terms of the flow. Abstractly, the cellular chain map $\partial : C_{k+1} \longrightarrow C_k$ is a homomorphism between free abelian groups with bases given by the critical points of index $k + 1, k$ respectively, and hence is given by a certain matrix with integer coefficients. We will see that these integers can be computed as intersection numbers (a geometric manifestation of Poincaré duality) of "in-spheres" of the index $k + 1$ critical points with the "out-spheres" of the index k critical points.

IV. Basis theorem. The group C_k is free abelian on the critical points of index k (in a way made precise in step III). Suppose we are given some other basis of C_k. The basis theorem says that provided $k \geq 2$, we can modify our Morse function so that the index k critical points of the new function correspond to the new basis (without changing anything else).

V. Cancellation Theorem A. Suppose f is a self-indexing Morse function on the triad W, with just two critical points of indices $k, k+1$, and suppose $H_*(W, V_0) = 0$. Then homologically the critical points cancel out. The *Cancellation Hypothesis* is the assumption that as the in-sphere of the top critical point flows down to the next level (going backwards in time), it intersects the outset of the lower critical point transversally in a single point. Assuming the Cancellation Hypothesis, we show that in this case the two critical points can indeed be "cancelled"; that is, f can be modified to produce a function with no critical points at all.

VI. Cancellation Theorem B. Unfortunately, verification of the Cancellation Hypothesis involves serious difficulties. By the flow-adjusting lemma and standard transversality theorems, we can easily arrange that the in-sphere of the top critical point has transverse intersection with the outset of the bottom critical point. Furthermore, the algebraic intersection number is ± 1 by purely homological considerations. But this does not mean the intersection consists of a single point; it only means that the number of intersection points is odd, and all but one cancel out in pairs of opposite sign. This lead to the general question of cancelling intersection points of opposite sign for a pair of transversally intersecting submanifolds. It turns out that this question is very subtle. It is answered affirmatively by a theorem of Whitney, but only with certain restrictions on the dimensions and on the fundamental groups. This is where the assumption "W is simply-connected of dimension at least six" comes in; under those hypotheses Whitney's theorem implies the Cancellation Hypothesis and we can cancel critical points as in /Step V.

VII. Low Index Theorem. At various points in the proof (e.g., the Basis theorem), critical points of low index would require special treatment. Provided that W is simply-connected and $n \geq 5$, the Low Index Theorem allows us to eliminate critical points of index zero and one. It follows that critical points of index n and $n-1$ can also be eliminated: Just turn the manifold upside down! (To be precise, replace f by $-f$; note this changes index k to index $n-k$.)

VIII. The h-cobordism theorem. With the above results in hand, the amazing proof can be roughly summarized as follows: Given a simply-connected h-cobordism W with $dim\, W \geq 6$, we wish to show that W admits a Morse function with no critical points. By the Rearrangement Theorem and Low-Index Theorem we can find a self-indexing Morse function f with no critical points of index $0, 1, n-1, n$. We then proceed by induction on the number of critical points. At the inductive step we will modify f to produce a new Morse function g with two fewer critical points, thereby completing the proof.

Let k be the minimal index occuring. Since W is an h-cobordism, $H_*(W, V_0) = 0$. Thus the cellular boundary map from the $(k+1)$-chains to the k-chains must be onto. So for any critical point p of index k (thought of as a basis element for the cellular k-chains), there must be some $(k+1)$-chain that hits it. Using the Basis Theorem, one can show that the $(k+1)$-chain can be taken to be a basis element corresponding to some critical point q of index $k+1$. Now use the level-adjusting lemma to bump p up and q down, producing a segment $f^{-1}[a, b]$ of W with only two critical points as in Cancellation Theorem A. Since the Cancellation Hypothesis holds by Cancellation Theorem B, we can cancel these out and we're done! Well...at least with the case $n \geq 6$. The case $n = 5$ requires some additional input, due to Kervaire-Milnor (to be discussed in due time).