1 Chain Complexes and Exact Sequences

A 3-term sequence of abelian groups \( A \xrightarrow{f} B \xrightarrow{g} C \) is said to be exact if the kernel of \( g \) coincides with the image of \( f \). An arbitrary sequence (finite or infinite) of abelian groups

\[
\ldots \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_{n+1} \rightarrow \ldots
\]

is exact if each sequence of three consecutive terms, as shown, is exact. A short exact sequence is an exact sequence of the form

\[
0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\]

Note that \( f \) is injective and \( g \) is surjective. The next result, known as the “5-lemma” is extremely useful, and elementary to prove.

**Lemma 1.1** Suppose given a commutative diagram of abelian groups

\[
\begin{array}{cccccc}
A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 & \longrightarrow & E_1 \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow i & & \downarrow j \\
A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & D_2 & \longrightarrow & E_2
\end{array}
\]
such that the rows are exact and \( f, g, i, j \) are isomorphisms. Then \( h \) is an isomorphism.

**Remark.** The proof actually yields a more refined statement, but this is the most commonly used version of the 5-lemma (prove it yourself!). Note also the special case in which the end-terms are zero groups so that the rows are short exact.

A *chain complex* \( C \) is a sequence of abelian groups and group homomorphisms

\[
\ldots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \ldots
\]

with the property that \( \partial_{n-1} \partial_n = 0 \). In general, the subscript \( n \) could range over all integers, but for present purposes we will take \( n \) nonnegative; equivalently, we assume that all \( C_n \) are zero for \( n \) negative. The kernel of \( \partial_n \) is called the group of \( n \)-cycles and denoted \( Z_nC \). The image of \( \partial_{n+1} \) is called the group of \( n \)-boundaries (or perhaps \( n+1 \)-boundaries; the literature is not consistent) and denoted \( B_nC \). Note that \( B_n \subset Z_n \). The quotient group \( Z_n/B_n \) is the *\( n \)-th homology group of \( C \)*, denoted \( H_nC \). Note that a long exact sequence is the same thing as a chain complex all of whose homology groups are zero.

Now observe that chain complexes form a category, in an obvious way. A map of chain complexes \( f : C \rightarrow D \) is simply a sequence of group homomorphisms \( f_n : C_n \rightarrow D_n \) that commutes with the boundary maps:

\[
\begin{array}{ccc}
C_n & \xrightarrow{f_n} & D_n \\
\downarrow{\partial_C} & & \downarrow{\partial_D} \\
C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1}
\end{array}
\]

Such a map necessarily takes cycles to cycles and boundaries to boundaries, and therefore induces a map on homology groups \( H_nC \rightarrow H_nD \), denoted \( f_* \) or \( H_nf \). It is then trivial to check that each homology group \( H_n \) defines a functor from the category of chain complexes to the category of abelian groups. Alternatively, we can assemble them all into a single graded
abelian group $H_\ast C = \bigoplus_{n=0}^{\infty} H_n C$, and regard $H_\ast$ as a functor to graded abelian groups.

A sequence of chain complexes is exact if, at each level $n$, it is exact as a sequence of abelian groups.

**Proposition 1.2** Suppose $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is a short exact sequence of chain complexes. Then there is a natural long exact sequence

$$\ldots \to H_n A \to H_n B \to H_n C \overset{\partial}{\to} H_{n-1} A \to \ldots$$

The homomorphism $\partial$ is defined as follows: Let $\gamma \in H_n C$ be represented by the cycle $c \in C_n$. By assumption $c = g_n(b)$ for some $b \in B_n$ (but $b$ need not be a cycle). Since $g$ is a map of chain complexes, $g_{n-1} \partial B b = 0$ and hence $\partial B b = f_{n-1}(a)$ for some $a \in A_{n-1}$. Since $f$ is a map of chain complexes, it follows that $\partial A a = 0$—i.e., $a$ is a cycle. Let $\alpha$ denote the homology class of $a$, and define $\partial \gamma = \alpha$. A somewhat long but easy exercise shows that $\partial$ is a well-defined homomorphism, and that the sequence of the proposition is exact.

One final remark: if $R$ is any ring, and we replace the term “abelian group” by “$R$-module” throughout, all definitions, results and proofs in this section go through verbatim. Thus we can talk about chain complexes of $R$-modules (the homology groups are then $R$-modules), exact sequences of $R$-modules, etc.

### 2 Definition of Singular Homology

The $n$-simplex $\Delta^n$ is the subset of $\mathbb{R}^{n+1}$ given by

$$\Delta^n = \{ (t_0, \ldots, t_n) : t_i \geq 0, \sum t_i = 1 \}$$

Thus $\Delta^0$ is a point, $\Delta^1$ is a line segment, $\Delta^2$ is a triangle, and so on. In general, $\Delta^n$ is homeomorphic to the $n$-disc. Note that the boundary of $\Delta^n$ is a union of $n+1$ copies of $\Delta^{n-1}$. There are canonical coface maps $\epsilon_i : \Delta^{n-1} \to \Delta^n$ defined by $\epsilon_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, 0, \ldots, t_n)$, where the zero is in the $i$-th slot.

Let $S_n X$ denote the set of singular $n$-simplices—that is, the set of all continuous maps $\Delta^n \to X$, and define face maps $d_i : S_n X \to S_{n-1} X$ by $d_i f =$
Let $C_nX$ denote the free abelian group $\mathbb{Z}S_nX$, and define boundary maps $\partial_n : C_nX \rightarrow C_{n-1}X$ by

$$\partial_n f = \sum_{i=0}^{n} (-1)^i d_i f$$

**Proposition 2.1** $\partial_{n-1} \partial_n = 0$.

The proof is an elementary combinatorial calculation. Hence $C_*X$ is a chain complex. The homology groups of this complex are the singular homology groups of $X$, denoted $H_*(X)$.

It is easy to see that the homology groups are covariant functors of $X$. In fact for each $n$, $X \mapsto S_nX$ is a functor from spaces to sets, with the induced maps given by composition. Explicitly, if $\phi : X \rightarrow Y$ is a continuous map, and $f : \Delta^n \rightarrow X$ is a singular $n$-simplex of $X$, $S_n(\phi)(f) = \phi f$. These functors are compatible with the face maps in an evident way, and we conclude that $X \mapsto C_*X$ is a functor from spaces to chain complexes. Composing with the homology functor from chain complexes to graded abelian groups shows that $X \mapsto H_*(X)$ is a functor from spaces to graded abelian groups. It follows immediately that homeomorphic spaces have isomorphic homology groups.

The proof of the next proposition is an easy exercise. The reason for calling it the Dimension Axiom need not concern us here.

**Proposition 2.2** Dimension Axiom.

Let $*$ denote the space consisting of a single point. Then $H_k(*)$ is isomorphic to $\mathbb{Z}$ if $k = 0$ and is zero otherwise.

Computing the homology groups of more interesting spaces $X$ requires the machinery of the following section. However we can get some information about low-dimensional homology groups here.

**Proposition 2.3** $H_0X$ is naturally isomorphic to the free abelian group on the set of path-components of $X$.

The proof is an important exercise. Much harder, although still “elementary” is:
Proposition 2.4 Let $X$ be a path-connected space with basepoint $x_0$. Then there is a natural homomorphism $h : \pi_1(X, x_0) \rightarrow H_1X$ that induces an isomorphism from the abelianization of $\pi_1(X, x_0)$ to $H_1X$.

The homomorphism $h$ is easy to describe. Represent an element of $\pi_1(X, x_0)$ by a path $\lambda : I \rightarrow X$ taking both endpoints to the basepoint. Then $\lambda$ can be regarded as a singular 1-simplex, and as such it is clearly a cycle. Thus $h$ is defined by taking the homotopy class of $\lambda$ to its homology class. One can show that this yields a well-defined surjective homomorphism, which then automatically factors through the abelianization $(\pi_1X)_{ab}$. The final and hardest step is to show that $(\pi_1X)_{ab} \rightarrow H_1X$ is injective.

3 Homotopy Invariance and the Mayer-Vietoris Sequence

Homotopy invariance and the Mayer-Vietoris sequence are the two key technical results needed to get the homology machine fired up. They are fairly easy to understand without knowing the proofs. In fact it is not hard to give the rough idea of the proofs, although this is best done in lecture.

Theorem 3.1 (Homotopy Invariance) Suppose the maps $f, g : X \rightarrow Y$ are homotopic. Then the induced maps on homology coincide: $H_*(f) = H_*(g)$.

It follows that if $f$ is a homotopy equivalence, then $H_*(f)$ is an isomorphism. In other words, homotopy equivalent spaces have isomorphic homology groups.

Now suppose $X$ is the union of two open subsets $U, V$. Let $j_U, j_V$ denote the inclusions of $U \cap V$ into $U, V$, respectively. Let $i_U, i_V$ denote the inclusions of $U, V$, respectively, into $X$. Finally let $j : H_*(U \cap V) \rightarrow H_*U \oplus H_*V$ have components $H_*(j_U), H_*(j_V)$ and let $i : H_*U \oplus H_*V \rightarrow H_*X$ have components $H_*(i_U), -H_*(i_V)$. Note the minus sign in the last definition.

Theorem 3.2 (Mayer-Vietoris sequence) Suppose $X$ is the union of two open subsets $U, V$. Then there are natural homomorphisms

$$\partial : H_nX \rightarrow H_{n-1}(U \cap V)$$
and a long exact sequence

\[\ldots \to H_n(U \cap V) \xrightarrow{i} H_n U \oplus H_n V \xrightarrow{j} H_n X \xrightarrow{\partial} H_{n-1}(U \cap V) \to \ldots\]

Note that if \(U \cap V\) is empty, the theorem implies that \(H_* X = H_* U \oplus H_* V\). This special case is easily proved directly, using the fact that a simplex is a connected space.

To prove Theorem 3.2, one might begin by writing down the sequence of chain complexes

\[0 \to C_*(U \cap V) \xrightarrow{j'} C_* U \oplus C_* V \xrightarrow{i'} C_* X \to 0\]

where \(j', i'\) are the chain level analogues of the homomorphisms \(j, i\) defined above. If this sequence was exact, the desired long exact sequence would be obtained immediately from Proposition 1.2. The problem is that \(i'\) need not be onto, since the image of a singular simplex \(\Delta^n \to X\) need not lie entirely in \(U\) or \(V\) (exactness at the other slots is easily checked). To get around this problem we need the Subdivision Lemma:

**Lemma 3.3** Let \(U\) be any open cover of the space \(X\), and let \(C^{U}_* X\) denote the subcomplex of \(C_* X\) generated by all singular simplices \(f : \Delta^n \to X\) having the property that \(f(\Delta^n) \subset U\) for some \(U \in U\). Then the inclusion map \(C^{U}_* X \to C_* X\) induces an isomorphism on homology groups.

Assuming the subdivision lemma - with \(U\) the two-element cover \(U, V\) - the proof of Theorem 3.2 is easy. Let \(H^{U}_* X\) denote the homology of \(C^{U}_* X\). Then by the subdivision lemma, \(H^{U}_* X \cong H_* X\), so in the statement of Theorem 3.2 we can replace \(H_* X\) by \(H^{U}_* X\). By definition there is a short exact sequence of chain complexes

\[0 \to C_*(U \cap V) \to C_* U \oplus C_* V \to C^{U}_* X \to 0\]

and hence a long exact sequence

\[\ldots \to H_n(U \cap V) \xrightarrow{i} H_n U \oplus H_n V \xrightarrow{j} H^{U}_n X \xrightarrow{\delta} H_{n-1}(U \cap V) \to \ldots\]
This proves the theorem. The proof of the subdivision lemma is rather long and technical, although the idea of the proof is fairly easy to explain (this is best done in lecture).

We can now compute the homology groups of the $n$-sphere. Note that the proof is axiomatic, in the sense that it proceeds by formally applying the Dimension Axiom, Homotopy Invariance, and the Mayer-Vietoris sequence.

**Theorem 3.4** If $n > 0$, $H_k S^n$ is isomorphic to $\mathbb{Z}$ for $k = 0, n$ and is zero otherwise. If $n = 0$, $H_k S^0$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ for $k = 0$ and is zero otherwise.

**Proof:** The proof is by induction on $n$. Since $S^0$ is the disjoint union of two points, the case $n = 0$ follows from the Dimension Axiom together with the simple consequence of the Mayer-Vietoris sequence mentioned above. Now suppose the theorem has been proved for $S^m$ with $m < n$, and let $U, V \subset S^n$ denote the complements of the two poles. Then $U$ and $V$ are contractible, so by homotopy invariance the terms $H_k U \oplus H_k V$ of the Mayer-Vietoris sequence are zero for $k > 0$. Hence the boundary map $\partial : H_k S^n \rightarrow H_{k-1}(U \cap V)$ is an isomorphism whenever $k > 1$. Since $U \cap V$ contains $S^{n-1}$ (the equator) as a deformation retract, for $k > 1$ the theorem follows by inductive hypothesis and another application of homotopy invariance. When $k = 1$ the Mayer-Vietoris sequence yields an exact sequence of the form

$$0 \rightarrow H_1 S^n \xrightarrow{\partial} H_0 S^{n-1} \xrightarrow{j} \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

where the last map takes $(a, b)$ to $a - b$. If $n > 1$ then $H_0 S^{n-1} \cong \mathbb{Z}$. This forces $j$ injective and $H_1 S^{n-1} = 0$. Finally, suppose $k = n = 1$ (this case follows from Proposition 2.4, of course, but we wish to emphasize the axiomatic approach here). Then $H_0 S^{n-1} \cong \mathbb{Z} \oplus \mathbb{Z}$. Since the image of $j$ has rank one, so does the kernel of $j$. Since $\partial$ is an isomorphism onto the kernel of $j$, this completes the proof.

We now have all the ingredients for some beautiful, spectacular applications. To appreciate the power of homology, you might try proving the Brouwer Fixed Point Theorem or the theorem on vector fields by some other method (good luck!).

**Theorem 3.5 Brouwer Fixed Point Theorem.**

Any continuous map from the $n$-disc $D^n$ to itself has a fixed point.
Proof: Let \( f : D^n \to D^n \) be a continuous map. We suppose that \( f \) has no fixed points and derive a contradiction. For each \( x \in D^n \), form the directed line segment from \( f(x) \) to \( x \) (this makes sense since \( f(x) \neq x \)). Extend this segment beyond \( x \) until it intersects \( S^{n-1} \), and call this intersection point \( g(x) \). Then \( g \) is a continuous map \( D^n \to S^{n-1} \) (exercise), and if \( x \in S^{n-1} \) then \( g(x) = x \). In other words, \( S^{n-1} \) is a retract of \( D^n \). Since homology is a functor, it follows formally that \( H_k S^{n-1} \) is a retract of \( H_k D^n \) for all \( k \). But \( D^n \) is contractible, so \( H_k D^n = 0 \) for \( k > 0 \) and \( H_0 D^n = \mathbb{Z} \). Taking \( k = n - 1 \) contradicts the preceding theorem.

The next application is to self-maps of \( S^n \). Suppose \( n > 0 \) and \( f : S^n \to S^n \). Then the induced map \( H_n f \) is an endomorphism of the free cyclic group \( H_n S^n \), and hence is multiplication by some integer \( d \). We call this integer the degree of \( f \), and write \( \deg f = d \). Note that homotopic maps have the same degree, by homotopy invariance. (In fact the converse is also true—maps of the same degree are homotopic—but this is much harder to prove.) Note also that \( \deg(fg) = \deg(f)\deg(g) \). In particular, if \( f \) is a homeomorphism then \( \deg f = \pm 1 \).

**Proposition 3.6** Suppose \( f(x) = Ax \) with \( A \in O(n) \). Then \( \deg f = \det A \).

The proof is an interesting, instructive exercise. Now recall the antipodal map \( T : S^n \to S^n \) is defined by \( T(x) = -x \).

**Theorem 3.7** The antipodal map of \( S^n \) is homotopic to the identity map if and only if \( n \) is odd.

**Proof:** One direction is elementary: If \( n \) is odd, say \( n = 2k - 1 \), then we can regard \( S^n \) as the unit sphere in \( \mathbb{C}^k \). Then \( H(x, t) = e^{2\pi it}x \) gives the desired homotopy.

On the other hand \( \deg T = \det T = (-1)^{n+1} \), by the proposition above. So if \( n \) is even, \( \deg T = -1 \), and since the identity has degree one the two maps cannot be homotopic.

**Theorem 3.8** \( S^n \) admits a continuous nonvanishing vector field if and only if \( n \) is odd.
Proof: A vector field on $S^n$ can be regarded as a continuous function $s : S^n \to \mathbb{R}^{n+1}$ such that $s(x)$ is orthogonal to $x$ for all $x$. If $s$ is nonvanishing, then dividing by the length we can assume $s$ is a function from $S^n$ to itself such that $s(x)$ is orthogonal to $x$ for all $x$. So we must show that such a function exists if and only if $n$ is odd.

If $n$ is odd, then as above we can regard $S^n$ as the unit sphere in some $\mathbb{C}^k$. In this case $s(x) = ix$ is a nonvanishing vector field.

Now suppose $n$ is even and that a nonvanishing vector field $s$ exists. Define a homotopy $H : S^n \times I \to S^n$ by $H(x, t) = (\cos \pi t)x + (\sin \pi t)s(x)$. Then $H$ is a homotopy from the identity to the antipodal map, contradiction.

Our next goal is to compute the homology of real and complex projective spaces. The method is the same in both cases, but we will do the complex case first because it is easier.

Theorem 3.9 $H_k\mathbb{C}P^n$ is isomorphic to $\mathbb{Z}$ if $k$ is even and $0 \leq k \leq 2n$, and is zero otherwise.

Proof: We proceed by induction on $n$. For $n = 0$ the theorem is just the Dimension Axiom, since $\mathbb{C}P^0 = \ast$. At the inductive step we use the Mayer-Vietoris sequence. Let $U$ denote the complement in $\mathbb{C}P^n$ of the point $[0, 0, ..., 0, 1]$. Let $V$ denote the complement of $\mathbb{C}P^{n-1}$, where $\mathbb{C}P^{n-1}$ is as usual embedded in $\mathbb{C}P^n$ using the first $n$ coordinates. Then $U$ contains $\mathbb{C}P^{n-1}$ as a deformation retract (exercise), $V$ is homeomorphic to $\mathbb{C}^n$ and so is contractible, and $U \cap V$ is homeomorphic to $\mathbb{C}^n - 0$ and so contains $S^{2n-1}$ as a deformation retract. Substituting these results into the Mayer-Vietoris sequence and making use of homotopy invariance, we find that the Mayer-Vietoris sequence can be rewritten as an exact sequence

$$... \rightarrow H_kS^{2n-1} \rightarrow H_k\mathbb{C}P^{n-1} \rightarrow H_k\mathbb{C}P^n \xrightarrow{\partial} H_{k-1}S^{2n-1} \rightarrow ...$$

If $k$ is not $2n-1$ or $2n$ the theorem follows immediately from the inductive hypothesis, since then the first and last groups displayed above are zero, and hence the middle map is a isomorphism. If $k = 2n - 1$ we find that $H_k\mathbb{C}P^n = 0$, and if $k = 2n$ we find that $\partial$ is an isomorphism. This completes the proof.
**Theorem 3.10** \( H_k \mathbb{R}P^n \) is isomorphic to \( \mathbb{Z} \) if either \( k = 0 \) or \( n \) is odd and \( k = n \), is isomorphic to \( \mathbb{Z}/2 \) if \( k \) is odd and \( 0 < k < n \), and is zero otherwise.

The proof begins exactly as in the complex case. We use induction on \( n \), and the Mayer-Vietoris sequence at the inductive step. The open sets \( U \) and \( V \) are defined in exactly the same way. As before, we have that \( U \) contains \( \mathbb{R}P^{n-1} \) as a deformation retract, \( V \) is homeomorphic to \( \mathbb{R}^n \) and so is contractible, and \( U \cap V \) contains \( S^{n-1} \) as a deformation retract. At the inductive step we have to separate into the even and odd cases. If \( n \) is odd, the result follows easily by inspecting the exact sequence. In the even case, however, we encounter something new. The problem arises in the part of the exact sequence involving \( H_{n-1} \). Here we find there is a short exact sequence

\[
0 \longrightarrow H_{n-1}S^{n-1} \xrightarrow{\pi_*} H_{n-1}\mathbb{R}P^{n-1} \longrightarrow H_{n-1}\mathbb{R}P^n \longrightarrow 0
\]

The first group is free cyclic, and so is the second by inductive hypothesis. But to finish the calculation we need to know that the image of \( \pi_* \) has index 2. If we can’t compute this map, the induction grinds to a halt. So we have to go back to the case \( n \) odd and prove the following lemma as part of the inductive step. At this point we have already proved that \( H_n \mathbb{R}P^n \cong \mathbb{Z} \) for this particular odd \( n \).

**Lemma 3.11** If \( n \) is odd and \( \pi : S^n \rightarrow \mathbb{R}P^n \) is the covering map, \( H_n \pi \) has index 2.

**Proof:** Let \( \tilde{U} = \pi^{-1}U \), etc. Then \( \tilde{U} \) is \( S^n \) with the two poles removed, \( \tilde{V} \) is the complement of the equator, and \( \tilde{U} \cap \tilde{V} \) contains the disjoint union of two copies of \( S^{n-1} \) as a deformation retract. By naturality of the Mayer-Vietoris sequence, there is a commutative diagram

\[
\begin{array}{ccc}
H_nS^n & \xrightarrow{\partial} & H_{n-1}(\tilde{U} \cap \tilde{V}) \\
\downarrow H_n\pi & & \downarrow H_{n-1}(\pi) \\
H_n\mathbb{R}P^n & \xrightarrow{} & H_{n-1}(U \cap V)
\end{array}
\]

where the rows are segments of the Mayer-Vietoris sequences for \( S^n \) and \( \mathbb{R}P^n \) respectively. Since the bottom arrow is an isomorphism, it is enough to
show that the composition $H_{n-1}(\pi)\partial$ (going around the righthand side of the square) has index 2. Now by naturality of the Mayer-Vietoris sequence, the top row is an exact sequence of $G$-modules, where $G$ is the group of order two acting on $S^n$ by the antipodal map $T$ as usual. It follows that we can choose $x \in H_{n-1}(\tilde{U} \cap \tilde{V})$ so that (i) $x, T_*x$ is a $\mathbb{Z}$-basis for $H_{n-1}(\tilde{U} \cap \tilde{V})$; and (ii) $\pi_*x$ generates $H_{n-1}(U \cap V)$. Since $n - 1$ is even, $x + T_*x$ maps to $x + -x = 0$ in $H_{n-1}U$. It follows that $x + T_*x$ generates the image of $\partial$. But $\pi_*x + \pi_*T_*x = 2\pi_*x$, completing the proof of the lemma.

With the lemma in hand, it’s all smooth sailing. The inductive step now goes through with no difficulties, proving the theorem.

4 Homology of Pairs and Excision

A pair of spaces $(X,A)$ consists of a space $X$ and a subspace $A$. Pairs of spaces form a category in an obvious way: a map of pairs $f : (X, A) : \rightarrow (Y, B)$ is just a continuous function $f : X \rightarrow Y$ such that $f(A) \subset B$. Then the singular chain complex of $A$ is a sub-chain complex of the singular chain complex of $X$: $C_*A \subset C_*X$. Let $C_*(X, A)$ denote the quotient chain complex $C_*X/C_*A$. The homology groups of the pair $(X, A)$, written $H_*(X, A)$ and also known as the relative homology groups, are the homology groups of $C_*(X, A)$.

By definition there is a short exact sequence of chain complexes

$$0 \rightarrow C_*A \rightarrow C_*X \rightarrow C_*(X, A) \rightarrow 0$$

Proposition 1.2 then yields at once the long exact sequence of a pair

$$\ldots \rightarrow H_nA \rightarrow H_nX \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}A \rightarrow \ldots$$

which is natural for maps of pairs.

As in the case of the Mayer-Vietoris sequence sequence, the long exact sequence of a pair is most often used to compute the homology of $X$, assuming that we know the homology of $A$ and the relative homology groups. But how can we compute the relative homology groups? As it stands, the only obvious approach is to use the same long exact sequence—but this is pointless if the homology groups of $X$ are unknown. What we need is an independent way to compute the relative homology groups. This is where excision comes in.
Theorem 4.1 Suppose $X$ is the union of two open sets $U, V$. Then the map of pairs $(U, U \cap V) \to (X, V)$ induces an isomorphism

$$H_*(U, U \cap V) \xrightarrow{\cong} H_*(X, V)$$

This isomorphism is called “excision” because we’ve “excised” the complement of $U$. Intuitively, the idea is simply that if we are going to kill off singular chains lying in $V$, we might as well just consider chains lying in $U$ to begin with. In fact, granting Lemma 3.3, the proof is easy: let $C_*^U X$ denote the subcomplex generated by simplices lying entirely in $U$ or $V$. Then the subdivision lemma plus the 5-lemma shows that $H_*(X, V) = H_*(C_*^U X / C_*^V)$. But clearly $C_*^U X / C_*^V = C_* U / C_* V$, and the result follows.

In fact excision and Mayer-Vietoris are equivalent, so for some purposes it doesn’t matter which one is used. On the other hand there are many situations where relative homology groups are indispensable, and excision is extremely useful. For example, we get a quick proof of Brouwer’s “invariance of domain” theorem:

Theorem 4.2 Suppose $W$ is a nonempty open set in $\mathbb{R}^m$, and $W'$ is an open subset of $\mathbb{R}^n$. If $W$ and $W'$ are homeomorphic, then $m = n$.

Proof: Let $f : W \to W'$ be a homeomorphism. Pick $x \in W$ and let $x' = f(x)$. Then $f$ induces a homeomorphism of pairs $(W, W - x) \to (W', W' - x')$ and hence induces an isomorphism on relative homology groups. Let $D$ be an open disc centered at $x$ and contained in $W$. Applying excision with $X = W, U = D, V = W - x$ shows that $H_*(W, W - x) \cong H_*(D, D - x)$. But $H_k(D, D - x)$ is $\mathbb{Z}$ for $k = m$ and zero otherwise. Since the same calculation applies to $(W', W' - x')$, this forces $m = n$.

This shows that the dimension of a topological manifold is uniquely defined. Another application is to the definition of orientability for topological manifolds. Here’s a brief sketch: Recall that one definition in the smooth case involves a certain 2-fold cover called the orientation bundle. To define it one first needs a notion of orientation at a point $x$ of the manifold. In the smooth case this was done using equivalence classes of bases for the tangent space at that point. For an arbitrary topological manifold $M$, this doesn’t make sense. Instead we proceed as follows. Suppose $M$ has dimension $n$. An
excision argument almost identical to the one above shows that \( H_k(M, M-x) \)
is a free cyclic group if \( k = n \) and zero otherwise. An orientation at \( x \) is thenjust a choice of generator of the free cyclic group \( H_n(M, M-x) \). From here
the definition proceeds as in the smooth case. As an instructive exercise, you
should show that if \( M \) is smooth, the new definition agrees with the old.

It is natural to ask whether \( H_\ast(X, A) \) is related to the homology of the
quotient space \( X/A \) in which \( A \) has been collapsed to a point. In general
there is no relation worth mentioning, since this quotient space can be very
badly behaved. For example, think of the case where \( X \) is a disc and \( A \) is
the complement of the center point; then \( X/A \) is a non-Hausdorff space with
two points. If \( A \) is a sufficiently nice closed subspace, however, the situation
tends to improve. In fact, it can be shown using excision that when \( X \) is a
CW-complex and \( A \) is a subcomplex, \( H_\ast(X, A) \cong H_\ast(X/A) \) for \( \ast > 0 \). (The
exception for \( \ast = 0 \) can be removed if we use “reduced” homology on the
right; see below.)

5 Refinements and Variations

5.1 Homology with coefficients

Recall that the singular chain complex was defined in terms of \( \mathbb{Z}S_nX \), the
free abelian group on the set of singular \( n \)-simplices \( S_nX \). Now an abelian
group is the same thing as a \( \mathbb{Z} \)-module. Replacing \( \mathbb{Z} \) by an arbitrary ring \( R \),
the entire construction goes through verbatim. Thus we define \( C_n(X; R) = RS_nX \),
the free \( R \)-module on the singular \( n \)-simplices of \( X \). The boundary
maps \( C_n(X; R) \rightarrow C_{n-1}(X; R) \) are defined exactly as before, yielding a chain
complex of \( R \)-modules. The homology groups of this chain complex are
denoted \( H_n(X; R) \): homology with coefficients in \( R \). Note that \( H_n(X; R) \) is
an \( R \)-module, and that \( H_n(X; \mathbb{Z}) \) is just \( H_nX \) as defined earlier.

Although the definition makes sense for any ring \( R \), in the beginning one
really only cares about a few special cases. The most important are \( \mathbb{Z}, \mathbb{Z}/p \)
\((p \text{ prime}), \mathbb{Q}, \mathbb{R}, \text{ and } \mathbb{C} \). For example, the case \( R = \mathbb{R} \) is important because
of a connection with DeRham cohomology. A famous theorem of DeRham
says that if \( M \) is any smooth manifold, the DeRham cohomology \( H^D_{pR}M \)
is isomorphic to \( \text{Hom}_\mathbb{R}(H_\ast(M; \mathbb{R}), \mathbb{R}) \), the linear dual of homology with real
coefficients. Similarly, the case \( R = \mathbb{C} \) arises in connection with complex
analytic analogues of DeRham cohomology.

It is probably not clear at first why homology with $\mathbb{Z}/p$ or $\mathbb{Q}$ coefficients would be of interest. In fact there is a simple utilitarian reason: $\mathbb{Z}/p$ and $\mathbb{Q}$ are fields, and therefore the corresponding homology groups are vector spaces. It is usually much easier to work with vector spaces rather than general abelian groups; for example, one can exploit dimension counting arguments. The reason for singling out the fields $\mathbb{Z}/p$ and $\mathbb{Q}$ is just that these are the prime fields. Furthermore, little information is lost in passing to field coefficients. It turns out that knowledge of $H_\ast(X; R)$ for $R = \mathbb{Z}/p$ ($p$ ranging over all primes) and $R = \mathbb{Q}$ is usually just as good as knowing $H_\ast(X; \mathbb{Z})$.

There is one subtle pitfall to beware of. Consider homology with $\mathbb{Z}/p$ coefficients. At the chain level we have $C_n(X; \mathbb{Z}/p) = (C_n X)/p$; in other words, the “mod $p$ chains” really are just the ordinary chains reduced mod $p$. This is not true for homology, however. Even though $H_\ast(X; \mathbb{Z}/p)$ is customarily called “mod $p$ homology”, it is not true in general that $H_\ast(X; \mathbb{Z}/p) = (H_\ast(X))/p$. This failure can be traced to the fact that reduction mod $p$ does not preserve monomorphisms of abelian groups (consider, for example, the homomorphism $p : \mathbb{Z} \to \mathbb{Z}$). The correct statement is the following. For an abelian group $A$, let $A[p] = \{a \in A : pa = 0\}$.

**Proposition 5.1** There is a short exact sequence

$$0 \to (H_n X)/p \to H_n(X; \mathbb{Z}/p) \to (H_{n-1} X)[p] \to 0.$$

**Corollary 5.2** If $H_\ast X$ has no $p$-torsion, then the naive formula $H_\ast(X; \mathbb{Z}/p) = (H_\ast(X))/p$ is valid.

In general one can think of $C_n(X; R)$ as $C_n X \otimes \mathbb{Z} R$. Then the pitfall can be expressed by saying that “tensoring with $R$ need not commute with homology”; in other words, $H_n(X; R)$ is not $H_n X \otimes R$, in general. Rational homology does not have this problem; one can show that $H_\ast(X; \mathbb{Q}) = H_\ast X \otimes \mathbb{Q}$ for all $X$. These ideas lie at the very beginnings of the subject of homological algebra.

**Remark:** It is also possible to define homology with coefficients in an abelian group $A$ (as opposed to a ring), as follows. Note that a free abelian group on a set $S$ is the same thing as the direct sum of copies of $\mathbb{Z}$ indexed by $S$. Saying it
this way suggests defining \( C_n(X; A) \) as the direct sum of copies of \( A \), indexed by the singular simplices \( S_n X \). Alternatively one could think in terms of tensor products, as in the preceding paragraph. The homology groups of this chain complex are denoted \( H_*(X; A) \). These groups are important for advanced topics—notably spectral sequences and obstruction theory—but need not concern us here.

5.2 Reduced homology

Consider the following assertions about homology groups:

- If \( X \) is contractible, the homology groups of \( X \) vanish—except for \( H_0 \).
- The homology groups of \( \mathbb{R} P^{2n} \) are 2-torsion groups—except for \( H_0 \).
- \( H_n S^n \cong \mathbb{Z} \)—except when \( n = 0 \).

The reduced homology groups provide a means of avoiding these annoying zero-dimensional exceptions.

Let \( * \) denote the space with a single point. Then any space \( X \) admits a unique map \( \epsilon : X \rightarrow * \), and we define the reduced homology groups \( \tilde{H}_* X \subset H_* X \) by \( \tilde{H}_* X = \text{Ker} (H_* \epsilon) \). By the Dimension Axiom, \( \tilde{H}_k X = H_k X \) if \( k > 0 \) and there is a short exact sequence

\[
0 \rightarrow \tilde{H}_0 X \rightarrow H_0 X \rightarrow \mathbb{Z} \rightarrow 0
\]

In particular, if \( X \) is path-connected then \( \tilde{H}_0 X = 0 \). Note that the three statements above can now be reformulated in terms of reduced homology, without having to make any exceptions for dimension zero: the reduced homology groups of a contractible space vanish; the reduced homology groups of \( \mathbb{R} P^{2n} \) are all torsion groups, and \( \tilde{H}_n S^n \cong \mathbb{Z} \) for all \( n \geq 0 \).

It is easy to see that the reduced homology groups define a functor from spaces to graded abelian groups, just as in the case of ordinary homology, and that this functor satisfies the homotopy axiom and has a Mayer-Vietoris sequence (if we had used reduced homology in our calculation of \( H_k S^n \), we would not have needed to consider the case \( k = 1 \) separately). The reduced analogue of the Dimension Axiom is the assertion \( \tilde{H}_0 S^0 \cong \mathbb{Z} \).
To repeat: Reduced homology is little more than a notational device that gets rid of the \( \mathbb{Z} \) common to \( H_0 \) of all nonempty spaces. There is nothing confusing about it, so don’t let it confuse you.