CW-complexes

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A *CW-complex* is first of all a Hausdorff space X equipped with a collection of *characteristic maps* $\phi^n_\alpha : D^n \longrightarrow X$. Here n ranges over the non-negative integers, D^n is the unit n-disc, and α ranges over some index set (or one index set for each n, if you prefer) that is customarily suppressed from the notation. These maps are subject to the following axioms:

A1. The restriction of each ϕ^n_α to the interior of D^n is an embedding.

A2. Let $e^n_\alpha = \phi^n_\alpha(Int\, D^n)$. Then as a set, X is the disjoint union of the e^n_α. (The e^n_α are called the *cells* of X.)

A3. $\phi^n_\alpha(S^{n-1})$ lies in a finite union of cells of lower dimension (i.e. of dimension $< n$).

A4. A subset W of X is closed if and only if $(\phi^n_\alpha)^{-1}W$ is closed in D^n for all n, α.

The term CW-complex comes from "closure-finite with the weak topology", where "closure-finite" refers to **A3** and "weak topology" refers to **A4**. A *finite complex* is a CW-complex with only finitely many cells. Observe that if X is a finite complex, **A4** is redundant, since W is the union of the compact sets $\phi^n_\alpha((\phi^n_\alpha)^{-1}W)$, and these are closed since X is Hausdorff.

If X has cells of dimension n but no cells of higher dimension, we say that X is n-dimensional. X is *infinite-dimensional* if there are cells of arbitrarily large dimension. A *subcomplex* A of X is a closed subspace which is a union of cells of X. It is clear that A is then itself a CW-complex, whose characteristic maps are just the given characteristic maps for those cells of X which lie in

A. The *n-skeleton* of X is the union of all the cells of dimension at most n. Notations for this vary; I'll denote it as X^n. Clearly the n-skeleton is a subcomplex. The *attaching map* ψ^n_α for e^n_α is the restriction of ϕ^n_α to S^{n-1}, regarded as a map $S^{n-1} \longrightarrow X^{n-1}$ (see **A3**).

Exercises. 1. A CW-complex is compact if and only if it is a finite complex.

2. The zero-skeleton of X is a discrete space.

1 Discussion of the weak topology axiom A4

Keep in mind that this section is relevant only for complexes with infinitely many cells.

Let $\bar{e}^n_\alpha$ denote the closure of e^n_α, and note that $\bar{e}^n_\alpha = \phi^n_\alpha(D^n)$. It follows that **A4** is equivalent to requiring that $W \subset X$ is closed if and only if $W \cap \bar{e}^n_\alpha$ is closed for all n, α, or that $W \subset X$ is closed if and only if the intersection of W with every finite subcomplex of X is closed.

Now let X be an arbitrary Hausdorff space (some of this works for non-Hausdorff spaces, but I want to avoid distracting technicalities). We say that the topology on X is *compactly generated* if it has the property that a subset W of X is closed if and only if its intersection with every compact subset of X is closed. From Exercise 1 and the above remarks, it follows that every CW-complex is compactly-generated. Any locally compact Hausdorff space is compactly-generated (exercise).

The class of compactly-generated Hausdorff spaces has some very convenient properties. For example, if X is such a space then a map $X \longrightarrow Y$ is continuous if and only if its restriction to every compact subspace is continuous. Often the latter condition is easier to check. Furthermore, for many purposes there is no loss of generality in supposing a space to be compactly-generated. For if X is any Hausdorff space, we can always re-topologize X by *declaring* a set to be closed if and only if its intersection with every compact subset is closed. Let κX denote X with this new topology. This is a finer topology, so the identity map $\kappa X \longrightarrow X$ is continuous. It is not a homeomorphism in general, but the distinction between the topologies is invisible to compact Hausdorff spaces W, in the sense that a map $W \longrightarrow X$ of sets is continuous in one topology if and only if it is continuous in the other topology. As an exercise, one can then easily check that $\kappa X \longrightarrow X$ induces an isomorphism on homology groups (essentially because a simplex is compact) and on homotopy groups[1](essentially because a sphere is compact). So as far as homology and homotopy groups are concerned, nothing is lost by replacing X with κX.

2 Examples of CW-complexes

Note that in the definition of CW-complex, the disc D^n could be replaced by any homeomorphic space, such as an n-cube or n-simplex.

Example 1. The real line admits the structure of 1-dimensional CW-complex with the integers as zero-cells and the intervals $[n, n+1]$ as 1-cells. More generally, $\mathbb{R}^n$ is an n-dimensional CW-complex in an obvious way: the n-cells are the cubes whose vertices have integer coordinates (so the 0-cells are these integer lattice points, etc.).

Example 2. For those who know the definiton of a simplicial complex, any simplicial complex is a CW-complex, whose n-cells are just the n-simplices.

Example 3. In the first two examples, the characteristic maps ϕ_α^n are embeddings. However, one of the advantages of CW-complexes is that the ϕ_α^n are *not* required to be embeddings on the boundary. In particular, S^n admits a very efficient CW-structure with just one 0-cell, one n-cell, and no other cells: take any point x (say the south pole) as 0-cell, and as characteristic map for the n-cell use a quotient map $\phi : D^n \longrightarrow S^n$ that takes the boundary of D^n to the 0-cell and is a homeomorphism from the interior to $S^n - x$.

Example 4. The torus has a CW-decomposition with one 0-cell, two 1-cells, and one 2-cell. If you think of the torus as a square with sides identified in the standard way, this decomposition should be obvious.

Example 5. $\mathbb{R}P^n$ has a CW-structure with one cell of dimension k, $0 \leq k \leq n$. To see this, identify D^k with the "northern hemisphere" $S^k_+ = \{(x_0, ..., x_k) \in S^k : x_k \geq 0)\}$ in S^k. Let $\mathbb{R}P^k \longrightarrow \mathbb{R}P^n$ denote the standard inclusion. Finally, let ϕ^k denote the composite $S^k_+ \longrightarrow \mathbb{R}P^k \longrightarrow \mathbb{R}P^n$, where the first map is the usual quotient map. Then it is easy to check that the

[1]Homotopy groups will be defined below.

ϕ^k are characteristic maps for a CW-structure on $\mathbb{R}P^n$. (When checking A1 and A2, observe at the same time that the cell e^k is precisely the set of points whose homogeneous coordinates have the form $[x_0, ..., x_k, 0, ...0]$ with $x_k \neq 0$,, and that such a point has unique homogeneous coordinates of the form $[a_0, ..., a_{k-1}, 1, 0, ..., 0]$.)

Example 6. $\mathbb{C}P^n$ has a CW-structure with one cell of dimension $2k$, $0 \leq k \leq n$. The construction and proof are exactly analogous to those of the previous example.

Example 7. The previous two examples can be generalized in a beautiful way to real and complex Grassman manifolds; see *Characteristic Classes* by J. Milnor. This CW-structure is called the *Schubert cell* decomposition. It has many surprising and beautiful connections with algebraic geometry, Lie theory, representation theory, combinatorics, etc.

Example 8. Any smooth manifold admits a CW-structure. In fact it is known that any smooth manifold can be triangulated, and hence admits the structure of a simplicial complex (see example 2).

Example 9. Suppose M is a compact smooth manifold and f is a Morse function on M. Then M is homotopy equivalent to a CW-complex having one k-cell for each critical point of f of index k. At first glance this might seem a weaker result than (8), since we only get the CW-structure up to homotopy-type. In fact the opposite is true, because the link between the CW-structure and the Morse function has powerful consequences. (And of course as far as homotopical and homological invariants are concerned, "up to homotopy-equivalence" is always good enough anyway.)

Example 10. Here is a typical example of an infinite-dimensional CW-complex: let $\mathbb{R}P^\infty$ denote the set of lines through the origin in $\mathbb{R}^\infty$ (a real vector space of countably infinite dimension, with basis $e_1, e_2, ...$). Regarding $\mathbb{R}^\infty$ as the ascending union of the $\mathbb{R}^n$'s, we see that as a set, $\mathbb{R}P^\infty$ is the ascending union of the $\mathbb{R}P^n$'s. The problem is how to topologize this union. We declare a set to be closed if and only if its intersection with every $\mathbb{R}P^n$ is closed, $n < \infty$. It is then immediate from Example 5 that $\mathbb{R}P^\infty$ is a CW-complex with one n-cell for each n, $0 \leq n < \infty$. Note that axiom A4 holds by construction.

3 Attaching cells

A CW-complex can and should be thought of as a space built up by "attaching cells". First consider an arbitrary space X and a map $f : S^n \longrightarrow X$. Form the quotient space $(D^{n+1} \coprod X)/\sim$, where the equivalence relation identifies $x \in S^n$ with $f(x) \in X$. This space is said to be obtained from X by attaching an $n+1$-cell via f, and is denoted $D^{n+1} \coprod_f X$. The map f is the attaching map for the cell. In categorical terms, the diagram

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & X \arrow[d, "i"] \\ D^{n+1} \arrow[r, "j"'] & D^{n+1} \coprod_f X \end{tikzcd}$$

is a pushout diagram of topological spaces. The maps i, j are the obvious maps coming from the definition.

Example 1. Suppose we want to build a torus by attaching cells. Start with a point (=0-cell), then attach two 1-cells, using the only possible attaching maps, to obtain $S^1 \vee S^1$, the wedge of two circles. Finally attach a 2-cell to $S^1 \vee S^1$ in the evident way suggested by viewing the torus as a quotient of a rectangle. Notice, by the way, that the attaching map for the 2-cell is precisely the commutator of the two generators of the free group $\pi_1(S^1 \vee S^1)$.

Example 2. Attach an n-cell to a point by the only possible attaching map. The result is an n-sphere (up to homeomorphism).

Example 3. Attach an $n+1$-cell to $\mathbb{R}P^n$ using the standard quotient map $\pi : S^n \longrightarrow \mathbb{R}P^n$. The resulting space is homeomorphic to $\mathbb{R}P^{n+1}$. (This follows easily from Example 5 of the previous section.)

The process of attaching cells can be generalized to allow more than one cell at a time. Suppose we are given maps $f^n_\alpha : S^{n-1} \longrightarrow W$ with α ranging over some index set (possibly infinite), and n is fixed. Then we can attach all the cells simultaneously by forming the pushout diagram

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_\alpha D^n \arrow[r] & Y \end{tikzcd}$$

Example 4. Let X be an arbitrary CW-complex. Then by definition, the $n+1$-skeleton is obtained from the n-skeleton by attaching $n+1$-cells - that is, there is a pushout diagram

$$\begin{tikzcd} \coprod S^n \arrow[r] \arrow[d] & X^n \arrow[d] \\ \coprod D^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

where the disjoint unions are over all $n+1$-cells of X.

So we can think of a CW-complex as a space obtained by starting from a discrete space (the 0-skeleton) and inductively attaching 1-cells, 2-cells, etc.

4 Homology of CW-complexes

Theorem 4.1 *Suppose Y is obtained from X by attaching n-cells, as in the preceeding section. Then the relative homology group $H_n(Y, X)$ is the free abelian group on the set of n-cells being attached. Furthermore $H_k(Y, X) = 0$ for $k \neq n$.*

The proof is an easy application of excision.

Corollary 4.2 *Suppose X is a CW-complex. Then $H_n(X^n, X^{n-1})$ is the free abelian group on the n-cells of X, and $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$.*

Now if X is a CW-complex, define the *n-th cellular chain group* by $C_n^{cell} X = H_n(X^n, X^{n-1})$. Define a boundary map $\partial : C_n^{cell} X \longrightarrow C_{n-1}^{cell} X$ as the composite

$$H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}X^{n-1} \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

where the first map is the boundary map in the long exact sequence of the pair (X^n, X^{n-1}) and the second map comes from the long exact sequence of the pair (X^{n-1}, X^{n-2}). It is clear that $\partial^2 = 0$, since ∂^2 is a composite of four maps, the middle two of which come from the long exact sequence of the same pair, namely (X^{n-1}, X^{n-2}). Thus $C_*^{cell}X$ is a chain complex - the *cellular chain complex* of X. The homology groups of this complex are the *cellular homology groups* of X, written $H_*^{cell}X$.

Theorem 4.3 *Let X be a CW-complex. Then the singular homology groups H_*X are isomorphic to the cellular homology groups $H_*^{cell}X$.*

Corollary 4.4 *If X has finitely-many n-cells, then H_nX is finitely-generated. In particular, the homology groups of a finite complex are finitely-generated.*

The proof, which is surprisingly easy, will be given in class. Note this shows in particular that the cellular homology groups are independent of the choice of CW-structure on a given space X. It will be clear from the proof that the isomorphism is natural with respect to *cellular* maps $f : X \longrightarrow Y$ between CW-complexes; that is, maps with the property that $f(X^n) \subset Y^n$ for all n.

There is a relative version of the theorem that applies to CW-pairs (X, A). Using it one can show:

Theorem 4.5 *Let (X, A) be a CW-pair. Then the map of pairs $(X, A) \longrightarrow (X/A, *)$ induces an isomorphism $H_*(X, A) \cong H_*(X/A, *) = \tilde{H}_*(X/A)$.*

Theorem 4.3 greatly simplifies many homology calculations. For example, suppose X is a path-connected CW-complex with only even-dimensional cells (e.g. $\mathbb{C}P^n$). Then the boundary maps in the cellular chain complex are all zero, for trivial reasons, and we conclude that H_nX is free abelian on the n-cells of X if n is even, and zero if n is odd.

For another example, consider once more the calculation of $H_*\mathbb{R}P^n$. The cellular chain complex takes the form

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \longrightarrow 0$$

where there is one copy of $\mathbb{Z}$ in each dimension from 0 to n. We conclude instantly from this that $H_k\mathbb{R}P^n$ is a cyclic group for all k and zero for $k > n$. To get the precise value of H_k we have to analyze the boundary maps more closely. This would make a very instructive and perhaps rather challenging exercise. (Since we have already computed the homology by other means, we could easily show the boundary maps above are forced by the known answer. The problem is to do the calculation directly from the cellular chain complex.)

Homology of CW-complexes can be characterized axiomatically:

Theorem 4.6 *Suppose given a sequence of functors G_n from the category of topological spaces to the category of abelian groups, $n \geq 0$, which sastisfy (1) Dimension axiom: $G_n(*)$ is $\mathbb{Z}$ for $n = 0$ and zero otherwise; (2) Homotopy invariance; (3) Mayer-Vietoris (or alternatively, excision, assuming relative G_n groups are defined). Then for all finite CW-complexes X there is a natural isomorphism $G_nX \cong H_nX$.*

If one assumes given a natural transformation $G_n \longrightarrow H_n$, inducing an isomorphism for $X = *$, the proof is an easy induction on the number of cells of X. To characterize homology on arbitrary CW-complexes, just one further axiom is needed; the slogan is "homology commutes with direct limits", but I won't try to describe it here. Further discussion can be found in the classic original source: the text by Eilenberg and Steenrod.

5 The Euler Characteristic

Let X be a finite CW-complex, of dimension n. Let a_i denote the number of i-cells of X. Then the *Euler characteristic* $\chi(X)$ (historically the first topological invariant ever considered!) is the alternating sum

$$\chi(X) = a_0 - a_1 + a_2 - .... + (-1)^n a_n$$

Now let b_k denote the rank of H_kX and let $\chi^H(X)$ denote the alternating sum of the b_k's:

$$\chi^H(X) = b_0 - b_1 + b_2 - .... + (-1)^n b_n$$

Theorem 5.1 $\chi(X) = \chi^H(X)$. *In particular, the Euler characteristic is independent of the choice of CW-structure on X, and is even a homotopy-type invariant of X.*

The proof is an exercise in rather trivial algebra. First show that if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of finitely-generated abelian groups, then $rank\, B = rank\, A + rank\, C$. Then restate the theorem as a purely algebraic result about chain complexes of finitely-generated abelian groups, of finite length, and prove it by induction on the length (=highest degree in which the chain complex is nonzero). If you know how to tensor with the rationals, the whole argument becomes elementary dimension-counting.

6 Homotopy Theory of CW-complexes

The purpose of this section is to outline some of the key theorems on the homotopy theory of CW-complexes (plus one theorem, the Hurewicz theorem, that applies to arbitrary spaces). The first result will seem rather obscure; it is included here only for future reference.

Theorem 6.1 Homotopy Extension Property: *Let (X, A) be a CW-pair, and let Y be any topological space. If $f : X \longrightarrow Y$ is any map, and $H : A \times I \longrightarrow Y$ is a homotopy with H_0 equal to the restriction of f to A, then H extends to a homotopy $X \times I \longrightarrow Y$.*

Now recall that a map $f : X \longrightarrow Y$ between CW-complexes is *cellular* if $f(X^n) \subset Y^n$.

Theorem 6.2 Cellular Approximation: *Any map $f : X \longrightarrow Y$ between CW-complexes is homotopic to a cellular map. Furthermore, if A is a subcomplex of X, and f is already cellular on A, the homotopy can be taken rel A.*

The power of this result is easy to see. For example, consider S^n with its minimal CW-structure as in example 3 of section 2. Then for $k < n$ the k-skeleton consists of only the basepoint. So by cellular approximation, we conclude that if X is any CW-complex of dimension less than n, any map $f : X \longrightarrow S^n$ is nullhomotopic. In particular this is true for $X = S^k$, $k < n$.

For another application, consider a CW-complex X and a map $S^n \longrightarrow X$. Let Y denote the space obtained by attaching an $(n+1)$-cell to X via f. We would like to say that Y is again a CW-complex, with just one new $(n+1)$-cell. If f maps S^n into the n-skeleton of X, this is true and easy to prove. In the general case this fails, but by cellular approximation we know f is *homotopic* to a map taking S^n to the n-skeleton. It follows that Y is at least homotopy-equivalent to a CW-complex of the desired type, using Lemma 3.6 on p. 20 of Milnor.

Now let X be any pointed space (i.e., a space equipped with a chosen basepoint). The *n-th homotopy group* of X, denoted $\pi_n X$, is the set of pointed homotopy classes of pointed maps $S^n \longrightarrow X$ (a basepoint for S^n having been fixed once and for all). For $n = 0$ this is just the set of path-components of X, and there is no natural group structure. For $n > 0$, however, there is a natural group structure defined as follows:

Identify S^n with $I^n/\partial I^n$, the n-cube with its boundary collapsed to a point. Then elements of $\pi_n X$ are maps from I^n to X that take the entire boundary to the basepoint, and similarly for homotopies. Given two such maps f, g, we define $f * g$ by $(f * g)(x_1, x_2, ..., x_n) = f(2x_1, x_2, ..., x_n)$ if $x_1 \leq 1/2$ and $(f * g)(x_1, x_2, ..., x_n) = g(2x_1 - 1, x_2, ..., x_n)$ if $x_1 \geq 1/2$. Notice that this is exactly the usual definition of the fundamental group when $n = 1$.

Theorem 6.3 *The product $f * g$ gives $\pi_n X$ a natural group structure for $n > 0$ (in particular, it is well-defined on homotopy classes, and defines a functor from the category of pointed spaces to the category of groups). If $n > 1$ this group structure is abelian (in which case we write $f + g$ in place of $f * g$). If X is path-connected, then up to non-natural isomorphism, $\pi_n X$ is independent of the choice of basepoint.*

A map of spaces $f : X \longrightarrow Y$ is a *weak equivalence* if for all choices of basepoint and all $n \geq 0$, the induced homomorphism $\pi_n f : \pi_n X \longrightarrow \pi_n Y$ is an isomorphism. If the spaces are path-connected, then we only need to consider a fixed choice of basepoint. Any homotopy equivalence is a weak equivalence, but the converse is false. The single most important property of CW-complexes is the following theorem of J.H.C. Whitehead (which probably motivated his definition in the first place).

Theorem 6.4 *Suppose X, Y are CW-complexes. Then any weak equivalence $f : X \longrightarrow Y$ is a homotopy equivalence.*

For example, if X is a CW-complex and all homotopy groups of X are zero (for π_0 this means X is path-connected), then X is contractible. Again, this is false for arbitrary spaces. In some cases, it is possible to show that a map induces an isomorphism on homotopy groups even without knowing explicitly what the homotopy groups are; then the theorem is useful as it stands. If, on the other hand, we want to show $\pi_* f$ is an isomorphism by explicit computation, the theorem is often useless, since computing the homotopy groups explicitly tends to be impossible. Fortunately, Whitehead also proved a weaker version based on homology:

Theorem 6.5 *Suppose X, Y are simply-connected CW-complexes. Then any map $f : X \longrightarrow Y$ inducing an isomorphism on all homology groups is a homotopy equivalence.*

The assumption that the spaces are simply-connected can be substantially weakened, but cannot be eliminated. For example, there exists a smooth compact 3-dimensional manifold with boundary (in particular a CW-complex) whose reduced homology groups all vanish, and yet has nontrivial fundamental group, so can't be contractible. The problem is that although any weak equivalence induces an isomorphism on homology groups (proof is hard), the converse isn't true without restrictions on the fundamental group. In the simply-connected case, however, every homology isomorphism is a weak equivalence (again, the proof is hard); so the second Whitehead theorem follows from the first.

Remark: Note that in both Whitehead theorems, the assumption that the spaces are CW-complexes could be replaced by the assumption that they are *CW-spaces*: that is, spaces having the homotopy type of a CW-complex.

Now fix once and for all a generator ι_n for $H_n S^n$. Given any space X and map $f : S^n \longrightarrow X$, we get a homology class $f_* \iota_n \in H_n X$. By homotopy invariance of homology, this class depends only on the homotopy class of the map f. In particular, having chosen a basepoint, we get a map of sets $h : \pi_n X \longrightarrow H_n X$ sending $[f]$ to $f_* \iota_n$. This map is called the *Hurewicz map*. In fact h is a group homomorphism; the proof of this makes an interesting exercise. When $n = 1$, this is the same map considered earlier in the notes on homology. In general, h is neither injective nor surjective. In fact, it is usually a long way from being an isomorphism; this is what makes homotopy theory so hard. In one case, however, we do get an isomorphism; this is the famous *Hurewicz theorem*:

Theorem 6.6 *Suppose $n > 1$ and $\pi_k X = 0$ for $k < n$. Then the Hurewicz map is an isomorphism $\pi_n X \cong H_n X$.*

As a corollary we get $\pi_n S^n \cong \mathbb{Z}$, since cellular approximation shows the lower homotopy groups are zero.