Homology of adjoint orbits via Morse theory

Let G be a Lie group with Lie algebra $\mathfrak{g}$. An *adjoint orbit* is an orbit of the natural action of G on $\mathfrak{g}$ induced by conjugation. For example, if $G = U(n)$ is the unitary group, with Lie algebra $\mathfrak{u}(n)$ the skew-Hermitian matrices, then the orbit of a diagonal matrix with two distinct entries is a complex Grassmannian. More generally, every "flag variety" can be obtained as an adjoint orbit. In these notes we use Morse theory to study the homology of adjoint orbits. Our inspiration is expository paper of Bott "The geometry and representation theory of compact Lie groups", found in the volume *Representation Theory of Lie Groups* edited by Atiyah et. al., although we pursue the matter in greater depth than is done there.

Some convenient notation: 1. If α is a smooth curve on a manifold M, defined on some neighborhood of 0 with $\alpha(0) = p$, then $\delta(\alpha) = \frac{d}{dt}|_{t=0}\alpha \in T_pM$.

2. If $v \in T_hG$ and $g \in G$, then $g_*v \in T_{gh}G$ is the image of v under left translation by g, and $vg_* \in T_{hg}G$ is the image of v under right translation.

1 The exponential map

Here we recall a few salient points; see [Lee], Chapter 20, for more background. Let G be a connected Lie group, with Lie algebra $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the tangent space at the identity element e, and can be identified with the left-invariant vector fields on G (thereby obtaining its Lie algebra structure). The *exponential map* $exp : \mathfrak{g} \longrightarrow G$ is defined as follows: Let $\theta_X(t)$ denote the integral curve of X through e. Then $exp\, X = \theta_X(1)$. Frequently, the following characterization is more convenient:

Proposition 1.1 *If $X \in \mathfrak{g}$ then the map $t \mapsto exp\, tX$ is the integral curve through e of the left-invariant vector field generated by X.*

Moreover $t \mapsto exp\, tX$ is a Lie group homomorphism $\mathbb{R} \longrightarrow G$ (a so-called "one-parameter subgroup").

Note that the proposition is not immediate; one has to check that $\theta_{tx}(1) = \theta_X(t)$. But this is not hard; see Lee, Proposition 20.5.

Some further noteworthy properties of *exp*:

- *exp* is smooth, and the differential $exp_*(0) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the identity. Hence *exp* maps some neighborhood of zero diffeomorphically onto a neighborhood of $e \in G$.
- *exp* is natural with respect to Lie group homomorphisms (see Lee, Prop. 20.8g).
- *exp* is a group homomorphism if and only if G is abelian.
- For $G = GL_n\mathbb{R}$ (or any closed Lie subgroup thereof), *exp* is just the classical exponential of matrices $exp(A) = e^A$.

We note also:

Proposition 1.2 *Let* $\theta : \mathbb{R} \times G \longrightarrow G$ *denote the flow associated to* X*. Then* $\theta(t, g) = g exp\, tX$*.*

Proof: Fix g and check the derivative.

Finally, here is one of the most striking applications of exp:

Theorem 1.3 *Let* G *be any Lie group,* H *a closed subgroup (not assumed to be even a submanifold). Then* H *is an embedded Lie subgroup.*

See Lee for the interesting and difficult proof.

2 The adjoint representation

Now G acts smoothly on itself by conjugation, with the identity element e as a fixed point. This yields a representation $Ad : G \longrightarrow GL(\mathfrak{g})$ called the *adjoint representation*. For GL_n or closed subgroups thereof it is just the conjugation action on matrices. It has a Lie algebra counterpart denoted lower-case $ad : \mathfrak{g} \longrightarrow End_{\mathbb{R}}\mathfrak{g}$, given by $ad(X)(Y) = [X, Y]$. Note ad is a homomorphism of Lie algebras. But there is another natural such Lie algebra homomorphism lying about, namely the differential Ad_* at the identity.

Proposition 2.1 $Ad_* = ad$*. Equivalently, for all* $X, Y \in \mathfrak{g}$ *we have*

$$\delta(Ad(exp(tX))Y) = [X, Y].$$

Proof: The two assertions are equivalent because one way to compute the value of a differential on a tangent vector v at point p is to take any smooth curve through p with tangent vector v, push it forward by the given map and take the derivative at 0. Here we use the curve $exp(tX)$.

To prove the displayed formula, recall that for any vector fields X, Y on a smooth manifold M, we have $[X, Y] = L_X Y$. Here $L_X Y$ is the Lie derivative, defined by

$$(L_X Y)_p = \delta(\theta_{-t*} Y_{\theta(t,p)}),$$

where θ is the flow associated to X. In our situation this yields

$$[X, Y] = \delta((exp\, tX)_* Y (exp\, - tX)_*) = \delta(Ad(exp\, tX)Y)$$

as desired.

3 Compact Lie groups

Now suppose G is compact. Then if $G \longrightarrow GL(V)$ is any representation, we can find find an inner product on V that is G-invariant, i.e. $\langle gv, gw \rangle = \langle v, w \rangle$, or equivalently $\langle gv, w \rangle = \langle v, g^{-1}w \rangle$, for all g, v, w. This is possible thanks to compactness: choose any inner product on V and average it over G. In particular there is an $Ad\, G$-invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$. Indeed the existence of such an inner product almost characterizes compact Lie groups: If there is an Ad-invariant inner product, then $G/C(G)$ is compact, where $C(G)$ is the center. To see this recall that any two inner products on V are equivalent under the action of $GL(V)$, from which it follows that $G/C(G)$ is conjugate in $GL(V)$ to a subgroup of the (compact) orthogonal group $O(V)$. The *Ad*-invariance is reflected in the Lie algebra as follows:

Proposition 3.1 *For all $X, Y, Z \in \mathfrak{g}$, we have*

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle.$$

Proof: We have

$$\langle Ad(exp(tY))X, Z \rangle = \langle X, Ad(exp(-tY))Z \rangle.$$

Applying $\frac{d}{dt}|_{t=0}$ to the left side yields $-\langle [X, Y], Z \rangle$; on the right side we get $-\langle X, [Y, Z] \rangle$.

A *torus* is a compact Lie group isomorphic to a product of circle groups. We can characterize such groups in several interesting ways.

Proposition 3.2 *Let G be a compact connected Lie group. Then G is abelian if and only if it is a torus.*

Proof: The "if" is immediate. Now suppose G is abelian. Then exp_G is a Lie group homomorphism whose image contains a neighborhood of the identity. But any neighborhood of the identity in a connected topological group generates the group (a standard exercise), so exp is surjective. It follows that $Ker\, exp$ is a discrete subgroup, free abelian with $rank = dim\, G$, and that $G \cong \mathbb{R}^n/\mathbb{Z}^n$, i.e. is a torus. Details of these last steps are left as an exercise.

A topological group H is *topologically cyclic* if it has a dense cyclic subgroup $< h >$; the generator h is called a *topological generator.*

Proposition 3.3 *Let G be a connected Lie group. Then G is topologically cyclic if and only if G is a torus.*

Proof: For the "if" one can assume $G = \mathbb{R}^n/\mathbb{Z}^n$, with $\mathfrak{g} = \mathbb{R}^n$. Then one shows that any $h = (x_1, ..., x_n)$ such that $1, x_1, ..., x_n$ are linearly over $\mathbb{Q}$ is a topological generator; this is a classical theorem of Kronecker. Note this shows further that the set of topological generators h is dense in $\mathfrak{g}$. For details see e.g. Brocker and tom Dieck, *Representation theory of compact Lie groups.*

Conversely suppose G is topologically cyclic, with topological generator g. Since $\langle g \rangle$ is dense and abelian, G is abelian (the fact that the closure of an abelian subgroup is abelian holds in any topological group whose identity element is closed, e.g. a Hausdorff group). If G is compact, then since G is connected it is a torus by what we proved earlier. This is all we really need, but for completeness we remark that a connected abelian Lie group is isomorphic to $T \times \mathbb{R}^k$ for some k, where T is a torus. For G topologically cyclic, it follows easily that $G = T$.

For another variant, note that the exponential map of a torus is surjective. So if $h \in T$ is a topological generator, there is an $X \in \mathfrak{t}$ such that $exp\, X = h$. By considering the corresponding one-parameter subgroup $exp\, tX$, we get the following generalization of Example 7.19 in Lee.

Corollary 3.4 *Let T be a torus. Then there exists $X \in \mathfrak{t}$ such that $exp\, tX$ is dense in T. In fact the set of such X is dense in $\mathfrak{t}$.*

Here is a further useful corollary, whose proof combines several of the ideas introduced so far:

Corollary 3.5 *Let G be a compact Lie group, $exp\, tX$ a one-parameter subgroup ($X \in \mathfrak{g}$). Then the closure $H := \overline{exp\, tX}$ is a torus, embedded as a closed Lie subgroup.*

Proof: Since H is a closed subgroup, it is a closed Lie subgroup. Since $exp\, tX$ is abelian, so is its closure. Since connectedness of subspaces is always preserved by closures, $exp\, tX$ is connected. So it is a compact connected abelian Lie group, hence a torus.

We conclude with a brief discussion of the representation theory of tori, leaving details to the reader. First of all, for any compact Lie group G and representation V of G, V splits as a direct sum of irreducible representations. To see this, we first choose a G-invariant inner product on V. Now if $W \subset V$ is any G-invariant subspace (meaning vector subspace), then the orthogonal complement of W is also G-invariant. Our assertion now follows by a trivial induction. In the case of tori we have further:

Proposition 3.6 *Let T be a torus. Then every non-trivial irreducible representation of T over $\mathbb{R}$ has dimension 2.*

Proof: Let V be a representation of T, and let h be a topological generator of T. By linear algebra there is an h-invariant subspace $W \subset V$ of dimension ≤ 2. By continuity plus density of $\langle h \rangle$, W is in fact T-invariant. If $dim\, W = 1$, then $T \longrightarrow GL(W) \cong \mathbb{R}^\times$ has image a compact 1-dimensional subgroup of $\mathbb{R}^\times$, hence is trivial. It follows that every non-trivial irreducible representation has dimension 2, as desired.

Thus every real representation of T splits as a direct sum of 2-dimensional irreducibles plus a summand on which T acts trivially.

4 Regular elements and Cartan subalgebras

Now let $X \in \mathfrak{g}$ and let $\mathfrak{g}_X$, $\mathfrak{g}^X$ denote respectively the kernel and image of $ad\, X : \mathfrak{g} \longrightarrow \mathfrak{g}$. Thus $\mathfrak{g}_X$ is the Lie centralizer of X and there is a short exact sequence

$$0 \longrightarrow \mathfrak{g}_X \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}^X \longrightarrow 0.$$

In fact the short exact sequence exists for any Lie algebra, but the following lemma is special to the compact setting.

Lemma 4.1 *There is an orthogonal direct sum decomposition*

$$\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X.$$

Moreover, it is a direct sum as $\mathfrak{g}_X$-modules, and $ad\, X$ is an isomorphism on $\mathfrak{g}^X$.

Proof: Suppose $[X, Y] = 0$. Then for all $Z \in \mathfrak{g}$ we have

$$\langle Y, [X, Z] \rangle = \langle [Y, X], Z \rangle = 0.$$

Hence $\mathfrak{g}_X \perp \mathfrak{g}^X$. In particular $\mathfrak{g}_X \cap \mathfrak{g}^X = 0$, so by the short exact sequence and dimension count we're done.

Remark: For general Lie algebras, $\mathfrak{g}_X \cap \mathfrak{g}^X$ can be nonzero. For example, Let $\mathfrak{g}$ have basis X, Y, Z with $[X, Y] = Z$ and all other brackets zero (i.e. the upper triangular 3 by 3 nilpotent matrices). Then $\mathfrak{g}_X = \mathbb{R}(X, Z)$ and $\mathfrak{g}^X = \mathbb{R}Z$.

An element $X \in \mathfrak{g}$ is *regular* if $\mathfrak{g}_X$ has minimal dimension, i.e. $dim\, \mathfrak{g}_X \leq \mathfrak{g}_Y$ for all Y. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h} = \mathfrak{g}_X$ for some regular X. Other versions of this somewhat obscure definition can be found in the literature; in the case of compact Lie algebras, however, there is a simpler characterization.

Lemma 4.2 *If $\mathfrak{h}$ is a Cartan subalgebra, then $\mathfrak{h}$ is a maximal abelian sublgebra. (The converse is true too, but the proof must wait.)*

Proof: Suppose $\mathfrak{h}$ is Cartan, so $\mathfrak{h} = \mathfrak{g}_X$ for some regular X. If $\mathfrak{h}$ is abelian then it is clearly maximal abelian. So suppose $\mathfrak{h}$ is not abelian. Then there exist Y, Z such that $[X, Y] = 0 = [X, Z]$ and $[Y, Z] \neq 0$. By Lemma 4.1, for all $t \in \mathbb{R}$, $ad\, X + tY$ preserves the decomposition $\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X$. Furthermore, since $ad\, X$ acts isomorphically on $\mathfrak{g}^X$ and the isomorphisms are an open subset of $End_{\mathbb{R}}\mathfrak{g}^X$, $ad\, X + tY$ also acts isomorphically for sufficiently small t. Hence $\mathfrak{g}_{X+tY} \subset \mathfrak{g}_X$. But $Z \notin \mathfrak{g}_{X+tY}$, contradicting $dim\, \mathfrak{g}_X$ minimal. Hence $\mathfrak{h}$ is maximal abelian as desired.

5 Morse functions on adjoint orbits

Let $\mathcal{O}$ be an orbit of the adjoint action of G on $\mathfrak{g}$, and $Y \in \mathcal{O}$.

Lemma 5.1 *$T_Y\mathcal{O} = \mathfrak{g}^Y$.*

Proof: T_Y is the image of $T_eG = \mathfrak{g}$ under the map $\phi : G \longrightarrow \mathfrak{g}$ given by $\phi(g) = Ad(g)Y$. Since $Ad_*(e) = ad$, we have $\phi_*(e)(Z) = [Z, Y]$ and the lemma follows.

Theorem 5.2 *Let $\mathcal{O}$ be an adjoint orbit. Let X be any regular element, with associated Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_X$. Let $f = f_X$ denote the height function $f(Y) = \langle X, Y\rangle$ on $\mathcal{O}$. Then f is a Morse function with $Crit\, f = \mathcal{O} \cap \mathfrak{h}$.*

Proof: We first show that $Crit\, f = \mathcal{O} \cap \mathfrak{h}$. By Lemma 5.1, Y is a critical point if and only if $\langle X, [Y, Z]\rangle = 0$ for all $Z \in \mathfrak{g}$, or equivalently $\langle [X, Y], Z\rangle = 0$ of all Z. Thus Y is critical if and only if $[X, Y] = 0$, i.e. $Y \in \mathfrak{h}$.

Next, we compute the Hessian at a critical point Y explicitly. Since $ad\,Y$ acts isomorphically on $\mathfrak{g}^Y = T_Y\mathcal{O}$, we can define $W^* = (ad\,Y)^{-1}W$ for $W \in T_Y\mathcal{O}$.

Lemma 5.3 *$H_f(V, W) = \langle X, [V, W^*]\rangle$.*

Proof: Recall that the Hessian is defined as follows: Choose a smooth function $\phi(s, t) : U \longrightarrow \mathcal{O}$ with U a neighborhood of the origin in $\mathbb{R}^2$, $\phi(0, 0) = Y$, $\frac{\partial\phi}{\partial s}(0, 0) = V$, $\frac{\partial\phi}{\partial t}(0, 0) = W$. Then

$$H_f(V, W) = \frac{\partial^2(f \circ \phi)}{\partial s \partial t}(0, 0).$$

This is a well-defined symmetric bilinear form, thanks to the fact that Y is a critical point. Here we take

$$\phi(s, t) = \exp(-sV^*) \exp(-tW^*)Y.$$

Then

$$\frac{\partial\phi}{\partial s}(0, 0) = \frac{d}{ds}|_{s=0} \exp(-sV^*)(z) = -[V^*, Y] = [Y, V^*] = V,$$

and similarly for W. By adjoint invariance of the inner product, we then have

$$f \circ \phi(s, t) = \langle \exp(sV^*)X, \exp(-tW^*)Y\rangle.$$

Applying $\partial^2/\partial s\partial t$ at the origin then yields

$$\langle [V^*, X], -[W^*, Y]\rangle = \langle [V^*, X], W\rangle = \langle -V^*, [W, X]\rangle = \langle [W, V^*], X\rangle,$$

proving the lemma.

Finally we show H_f is nondegenerate. Suppose $\langle [W, V^*], X\rangle = 0$ for all $W \in \mathfrak{g}^Y$. Then $\langle [W, V^*], X\rangle = \langle W, [V^*, X]\rangle$, and since $[V^*, X] \in \mathfrak{g}^Y$ we conclude that $[V^*, X] = 0$. Moreover, $V^* \in \mathfrak{g}_X \subset \mathfrak{g}_Y$, since $Y \in \mathfrak{g}_X$. Hence $V^* \in \mathfrak{g}_Y \cap \mathfrak{g}^Y = 0$. This completes the proof of the theorem.

Corollary 5.4 *For any adjoint orbit $\mathcal{O}$ and Cartan subalgebra $\mathfrak{h}$, $\mathcal{O} \cap \mathfrak{h}$ is finite and nonempty.*

That the intersection is nonempty follows from the compactness of $\mathcal{O}$, since f must then have at least one critical point. This yields further corollaries:

Corollary 5.5 *a) Any two Cartan subalgebras are conjugate under the adjoint action of G.*

b) Every maximal abelian subalgebra is a Cartan subalgebra (so any two such are conjugate).

c) Any two maximal tori of G are conjugate.

Proof: a) Let $\mathfrak{h}_1, \mathfrak{h}_2$ be Cartan subalgebras, X_1 a regular element of $\mathfrak{h}_1$. Then by the previous corollary, $gX_1 \in \mathfrak{h}_2$ for some g. Then $\mathfrak{h}_2 \subset \mathfrak{g}_{gX_1} = g\mathfrak{g}_{X_1} = g\mathfrak{h}_1$, and since $\mathfrak{h}_2$ is maximal abelian, the inclusion is an equality.

b) Let $\mathfrak{t}$ be a maximal abelian subalgebra; we know that $\mathfrak{t}$ is the Lie algebra of a maximal torus T; choose $Y \in \mathfrak{t}$, so that $exp\, tY$ is dense in T. Let $\mathfrak{h} = \mathfrak{g}_X$ be a Cartan subalgebra, X regular. Since $gY \in \mathfrak{h}$ for some g by Corollary 5.4, we can assume $Y \in \mathfrak{h}$. Now $\mathfrak{h}$ is maximal abelian, so is the Lie algebra of a maximal torus H. Then $exp\, tY \in H$ for all t, so $T \subset H$. Since T is maximal, $T = H$ and hence $\mathfrak{t} = \mathfrak{h}$ is a Cartan subalgebra.

c)b) One can show that a torus $T \subset G$ is maximal if and only if its Lie algebra $\mathfrak{t}$ is a maximal abelian (hence Cartan) subalgebra. Then (b) follows from (a). Details are left to the reader.

Here's an exercise that clarifies the relationship between regular X and X such that $exp\, X$ is a topological generator of T.

Exercise: a) Suppose T is a maximal torus, $X \in \mathfrak{t}$ and $exp\, X$ is a topological generator of T. Then X is regular.

b) Give a counterexample showing the converse is false. Even better, find a good theorem of the form "X is a topological generator if and only if X is regular *and...*"

We next compute the indices of the critical points. First observe that as T-modules there is a splitting $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_\theta \mathfrak{g}_\theta)$, where the $\mathfrak{g}_\theta$'s are irreducible 2-dimensional representations of T. Thus $\mathfrak{h}$ acts on $\mathfrak{g}_\theta$ via skew-adjoint transformations, and since the skew-adjoint transformations of a 2-dimensional inner product space have dimension 1, $\mathfrak{so}_\theta$ is generated by the image of $ad\, X$. Denote this image X_θ.

Now let $Y \in \mathfrak{h}$ be a critical point. Then the decomposition $\mathfrak{g} = \mathfrak{g}_Y \oplus \mathfrak{g}^Y$ is clearly compatible with the above root space decomposition; in fact

$$\mathfrak{g}^Y = \oplus_{[Y,\mathfrak{g}_\theta]\neq 0} \mathfrak{g}_\theta.$$

Thus the image of Y in $\mathfrak{so}_\theta$ is a multiple of X_θ; i.e. there is a $c_\theta \neq 0$ for all $Z \in \mathfrak{g}_\theta$ we have $[Y, Z] = c_\theta [X, Z]$. Let $m_Y = |\theta : c_\theta > 0|$.

Proposition 5.6 *Let $Y \in \mathcal{O}$ be a critical point. Then $\iota_Y = 2m_Y$.*

Proof: Consider the restriction of the Hessian quadratic form Q to $\mathfrak{g}_\theta \subset T_Y\mathcal{O}$. We have

$$Q(V) = \langle X, [V, V^*]\rangle = -\langle X, [[V^*, [Y, V^*]]\rangle = -\langle [X, V^*], [Y, V^*]\rangle = -c_\theta |[X, V^*]|^2.$$

Hence $\oplus_{c_\theta > 0}\mathfrak{g}_\theta$ is a maximal negative definite subspace for Q, proving the proposition.

Finding the critical points explicitly in specific cases is easy, using the Weyl group $W = N_G T/T$. For example, when $G = U(n)$, $W = S_n$, the symmetric group on n letters (identified with the permutation matrices).

Proposition 5.7 *$\mathcal{O} \cap \mathfrak{h}$ is an orbit of the Weyl group.*

Proof: Suppose $Y, Z \in \mathfrak{h}$, and $gY = Z$. Then $\mathfrak{h}, g\mathfrak{h} \subset \mathfrak{g}_Z$. So by Corollary 5.5, there is an element $x \in G_Z^0$ such that $xg\mathfrak{h} = \mathfrak{h}$, so $xg \in N_G T$. But $xgY = xZ = Z$, QED.

As a corollary (which can also be proved directly) we get:

Corollary 5.8 *W is finite.*

Proof: Take X so that $exp\, X$ is a topological generator of T (hence X regular). If $g \in N_G T$ and $gX = X$, then g centralizes $exp\, tX$ for all t and hence centralizes T. So $g \in C_G T$. Thus $N_G T/C_G T \cong \mathcal{O}$ is finite. Now observe that $C_G T/T$ is zero-dimensional, since T is a maximal torus. Since it is also compact, it is finite. (Here $C_G T$ is the centralizer.) This proves the result.

Remark: In fact one can show that $C_G T = T$.

6 Some consequences

We can now instantly compute the Euler characteristic χ and indeed the total rank of the homology.

Proposition 6.1 *Let $\mathcal{O}_Y$ be an adjoint orbit as above. Then*
a) $H_*\mathcal{O}_Y$ is free abelian and concentrated in even dimensions.
b) $\chi(\mathcal{O}_Y) = rank\, H_*\mathcal{O}_Y = |W/W_Y|$.

Proof: Since the critical points all have even index, an adjoint orbit $\mathcal{O}$ is homotopy-equivalent to a finite complex with cells in even dimensions. Moreover the cells are in bijective correspondence with W/W_Y. All of the assertions of the proposition then follow immediately.

Corollary 6.2 *Let T be a maximal torus of G. Then $\chi(G/T) = rank\, H_*G/T = |W|$.*

Remark: One of the key theorems in the structure theory of connected compact Lie groups states that (i) any two maximal tori are conjugate; and (ii) every element lies in a torus (hence in a maximal one). We have already proved (i) above, but we have not proved (ii). Note that (ii) is equivalent to saying that for any $g \in G$ and given maximal torus T, the action of g on G/T has a fixed point. One elegant proof of (ii) then proceeds by applying the Lefschetz Fixed-Point Theorem from algebraic topology; this requires knowing that $\chi(G/T) \neq 0$. For those familiar with the Lefschetz theorem, here's the proof of (ii): Let $\phi_g : G/T \longrightarrow G/T$ be the left action of g. By the Lefschetz theorem, it suffices to show the Lefschetz number $L(g)$ is nonzero. Since G is path-connected, ϕ_g is homotopic to the identity. So $L(g) = L(Id) = \chi(G/T) = |W| \neq 0$, QED!

We also remark that it is easy to show that (ii) is equivalent to exp_G being surjective, but a direct proof that exp is surjective is not so easy.

We can do much better than the Euler characteristic, however, by computing the ranks of the individual homology groups. This information is best encoded in the Poincaré polynomial, which is defined for any finite complex X (or space homotopy-equivalent to such a complex) by

$$|X|(t) = \sum_j (rank\, H_j X) t^j.$$

In our situation $H_j X$ will be zero for all odd j, so it's a bit easier on the brain to assign dimension 2 to the indeterminate t and write instead

$$|X|(t) = \sum_j (rank\, H_{2j} X) t^j.$$

For example, in this new notation $\mathbb{C}P^2$ has Poincaré polynomial $1 + t + t^2$, whereas in the old it would be $1 + t^2 + t^4$. Now for $Y \in \mathfrak{h}$, define

$$p_Y(t) = \sum_w t^{m_w},$$

where w ranges over a set of coset representatives of W/W_Y and $m_w = m_{wY}$ (the factor of 2 occuring in the index formula above is omitted because of our new convention). We then have at once:

Proposition 6.3 *Let $\mathcal{O}_Y$ be an adjoint orbit, $Y \in \mathfrak{h}$. Then*

$$|\mathcal{O}_Y| = p_Y.$$

7 Examples

Example 1. Let $G = U(n)$ be the unitary group, with Lie algebra $\mathfrak{g}$ the skew-Hermitian matrices. The diagonal matrices form a maximal torus T, whose Lie algebra consists of diagonal matrices with pure imaginary entries, and is our Cartan subalgebra $\mathfrak{h}$. The Weyl group W is just the symmetric group S_n, acting in the evident way. As regular element X we choose a decreasing sequence $a_1 > a_2 ... > a_n$ and take $X = diag(a_1 i, ..., a_n i)$.

Now consider the orbit $\mathcal{O}$ of $Y = diag(i, ...i, -i, ..., -i)$, where there are k i's and $n-k$ $-i$'s. Clearly $G_Y = U(k) \times U(n-k)$, and $\mathcal{O}$ is the Grassmann manifold $G_k\mathbb{C}^n$. Similarly $W_Y = S_k \times S_{n-k}$. At this point we can already conclude:

Proposition 7.1 $\chi(G_k\mathbb{C}^n) = rank\, H_*G_k\mathbb{C}^n = \binom{n}{k}$.

But we can do much better: we compute the Poincaré polynomial explicitly. There is a canonical set of coset representatives for $S_n/(S_k \times S_{n-k})$, namely the *shuffles*: the permutations w such that the restriction of w to $\{1, ..., k\}$ and $\{k+1, ..., n\}$ is order-preserving. Note that a shuffle is uniquely determined by its *Schubert symbol* $\sigma = (\sigma_1 < ... < \sigma_k)$, where $\sigma_i = w(i)$. From now on we identify shuffles with their Schubert symbols.

Now for $r < s$ let $M_{r,s} \subset \mathfrak{g}$ denote the skew-Hermitian matrices whose (a, b)-coordinate is zero if (a, b) is not (r, s) or (s, r). Note $dim_{\mathbb{R}} M_{r,s} = 2$, and that it is invariant under the actions of T and $\mathfrak{h}$. Moreover the action is determined by what it does on the above diagonal coordinate, so for present purposes the below-diagonal part can be ignored. Thus $[X, -]$ multiplies $M_{r,s}$ by $(a_r - a_s)i$ (where $i = \sqrt{-1}$). If w is a shuffle with Schubert symbol σ, it follows that

$$m_w = d(\sigma) := \sum_{j=1}^{k} (\sigma_j - j).$$

(Note $\sigma_j - j$ is the number of $-j$'s to the left of the $+j$ that has moved to slot σ_j. Thus the righthand side of the displayed equation is the number of root spaces on which wY acts as a negative multiple of X, as required.)

This reduces the computation of the Poincaré polynomial to a purely combinatorial problem. The following are polynomial analogues of n, $n!$, and $\binom{n}{k}$ respectively:

$$[n](t) = \frac{1-t^n}{1-t} = 1 + t + ... + t^{n-1}$$

$$[n!](t) = \prod_{i=1}^{n} [i](t)$$

$$[\binom{n}{k}](t) = \frac{[n!](t)}{[k!](t)[(n-k)!](t)}.$$

These are all polynomials with non-negative integer coefficients. In the case of $\binom{[n}{k]}$ this follows from the Pascal's triangle recursion formula

$$t^{n-k}[\binom{n}{k}] + [\binom{n}{k+1}] = [\binom{n+1}{k+1}],$$

whose proof we leave to the reader). Note that when evaluated at $t = 1$ these yield n, $n!$, $\binom{n}{k}$ respectively.

Proposition 7.2 $|G_k\mathbb{C}^n| = [\binom{n}{k}]$.

Proof: We know that $|G_k\mathbb{C}^n| = p_{k,n} := \sum_\sigma t^{m_\sigma}$, where σ ranges over the k-shuffles and m_σ is computed as above. It is easy to check that $p_{k,n}$ satisfies the same Pascal recursion formula as the $[\binom{n}{k}]$'s, with the same initial conditions.

For a specific example, we find that $G_2\mathbb{C}^4$ has Poincaré polynomial $1+t+t^2+t^3+t^4$ with our $|t|=2$ convention. Thus the "real" Poincaré polynomial is $1+t^2+t^4+t^6+t^8$.

Remark: The Grassmannian can also be realized as a homogeneous space $GL_n\mathbb{C}/P_k$, where P_k is the evident block triangular subgroup of matrices preserving $\mathbb{C}^k$. This point of view leads to the conclusion that $G_k\mathbb{C}^n$ is a projective algebraic variety; it is an example of a class of particularly beautiful varieties known as *flag varieties.*

I'll briefly sketch the remaining examples. Working out the details is a highly recommended exercise.

Example 2. Again take $G = U(n)$, X as above, but now let $Y = X$. Then $G_Y = T$ and $\mathcal{O} = U(n)/T$ is the space of ordered n-tuples of orthogonal lines in $\mathbb{C}^n$. Clearly $W_Y = S_n$.

Proposition 7.3 $|U(n)/T| = [n!](t)$. *In particular,* $\chi(U(n)/T) = rank\, H_*U(n)/T = n!$.

The proof is analogous to the Grassmannian case, and in some ways easier. One arrives at the combinatorial problem of counting pairs $i<j$ such that $\sigma(i) > \sigma(j)$, for given $\sigma \in S_n$, which one can manage by inductive arguments. (On the other hand, the Euler characteristic formula is again immediate.)

Remark: Here again we have a complex projective variety: $U(n)/T = GL_n\mathbb{C}/B$, where B is the "Borel subgroup" of upper triangular matrices.

In the next two examples, $G = SO(n)$. Here the real Grassmannians are *not* adjoint orbits (this would contradict our main theorem, since their homology is not concentrated in even dimensions—think of real projective space, for example).[1] We also mention that from a Lie-theoretic point of view, the infinite family of groups $SO(n)$ really falls into two separate families: n even and n odd. An thorough explanation of this point would require too long a digression into "root systems" and the classification of compact Lie groups, but this is what lurks behind the parity distinctions that you'll find below.

Recall that the Lie algebra of $SO(n)$ is $\mathfrak{so}(n)$, the skew-symmetric $n \times n$ matrices. As Cartan subalgebra $\mathfrak{h}$ we can take the 2×2 block-diagonal matrices, where for n odd there will be an extra zero in the nn position. For $a \in \mathbb{R}$, let $\overline{a}$ denote the skew-symmetric matrix

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

For $n = 2m$ or $2m+1$ we then we identify $\cong \mathfrak{h}$ by identifying $(a_1, ..., a_m)$ with the evident block diagonal matrix $diag(\overline{a_1}, ..., \overline{a_m})$ (with that extra zero in the lower right corner when n is odd). In this notation the Weyl group can be described as follows: For $n = 2m$, it is

[1] They do arise from a variant of the adjoint orbit construction, but we won't go into that here.

the group of order $2^{m-1}m!$ generated by permutations of the a_i's and multiplication of an even number of entries by -1. For $n = 2m+1$ it is the group of order $2^m m!$ defined in the same way but without the even parity restriction. As a regular element we may take any $X = (a_1, ..., a_m)$ with $a_1 > a_2 > ... > a_m > 0$.

Example/Exercise 3. Let $n = 2m$ or $n = 2m+1$ and $Y = (1, 0, ..., 0)$. Then the orbit $\mathcal{O}_Y$ is $SO(n)/S(O(2) \times SO(n-2))$, which is the Grassmanian $\tilde{G}_2\mathbb{R}^n$ of *oriented* 2-planes in $\mathbb{R}^n$. The W-orbit of Y consists of elements with one non-zero entry ± 1, so in both cases we have at once that the Euler characteristic is $2m$. The homology is the same as for $\mathbb{C}P^{2m-1}$ in the odd case, and the same as for $\mathbb{C}P^{2m-2}$ in the even case except that the middle dimension H_{2m-2} has rank 2.

Show also that $\mathcal{O}_Y$ is diffeomorphic to the quadric hypersurface $\sum z_i^2 = 0$ in $\mathbb{C}P^{n-1}$.

Example/Project 4. Now consider $Y = (1, 1, ..., 1)$ in the above notation. Then the isotropy group of Y is $U(m) \subset SO(2m+1)$, so $\mathcal{O}_Y = SO(n)/U(m)$. If you check the Weyl group action, you'll find that $\chi(\mathcal{O}_Y)$ is 2^{m-1} in the even case and 2^m in the odd case. Computing the Poincaré polynomial is again a combinatorial exercise, a bit harder this time.

These orbits have an interesting interpretation as projective varieties. For a quadratic form Q on a vector space V (real or complex), a subspace on which Q vanishes identically is called an *isotropic space*. For suitable k one can then define *isotropic Grassmannian* $G_k(V, Q) \subset G_k V$ as the subspace of k-dimensional Q-isotropic spaces. Now take $V = \mathbb{C}^n$, where again $n = 2m$ or $n = 2m+1$, and let $Q = \sum z_i^2$. If $k = 1$ then $G_k(\mathbb{C}^n, Q)$ is precisely the quadric hypersurface of Example 3. At the opposite extreme, it is not hard to show that maximal Q-isotropic spaces have dimension m. For n odd there is a natural transitive action of $SO(n)$ on $G_m(\mathbb{C}^n, Q)$, with isotropy group $U(m)$. So the Grassmannian of maximal isotropic spaces is an adjoint orbit. For n even, $O(n)$ acts transitively on $G_m(\mathbb{C}^n, Q)$ with isotropy group $U(m)$. Thus there are two path-components, with $SO(n)$ acting transitively on each, again with isotropy group $U(m)$. So again we have an adjoint orbit.

Working out the details of these assertions is a lengthy but rewarding project.