Wednesday 4/30 lecture

March 29, 2016

A topic left over from last quarter: The action of the fundamental groupoid on homotopy classes of maps, local coefficient systems and related facts.

1 Actions of the fundamental groupoid

Let X be any space. The fundamental groupoid \( \Theta(X) \) is the category whose objects are the points of X and whose morphisms \( x_0 \to x_1 \) are path-homotopy classes of paths \( \lambda : x_0 \to x_1 \). Composition is given by concatenation of paths. The identities are the constant paths, and since every path has an inverse, every morphism is an isomorphism. Thus \( \Theta(X) \) is a groupoid.

One minor complication is that composition occurs in the opposite order from concatenation: If \( \mu : x_1 \to x_2 \) is another path, then \( \mu \circ \lambda = \lambda \ast \mu \). In particular, the automorphism group of an object \( x_0 \) is technically not \( \pi_1(X, x_0) \) but rather its opposite group. Of course, every group is isomorphic to its opposite via \( g \mapsto g^{-1} \), but still we have to manage the reversal of order. A simple and painless way to do this is modelled on group actions. If \( G \) is a group, thought of as a groupoid with one object, then left (resp. right) \( G \)-actions correspond to covariant (resp. contravariant) functors. By putting the “action” of the fundamental groupoid on the right, the reversal of order is automatically accounted for. We illustrate this vague statement with a familiar example that motivates all of the “groupoid actions” considered here.

For each \( x_0 \in X \), we have the fundamental group \( \pi_1(X, x_0) \) based at \( x_0 \). Moreover any path \( \lambda : x_0 \to x_1 \) defines an isomorphism \( \pi_1(X, x_0) \to \pi_1(X, x_1) \), via \( \alpha \mapsto \lambda^{-1} \ast \alpha \ast \lambda \). Of course one has to check that this is well-defined on the homotopy classes involved. The result is a functor from \( \Theta(X) \) to groups, but it is a contravariant functor. I find it easier (this is just a personal preference) to use the notation and terminology of group actions, writing \( \alpha \cdot \lambda := \lambda^{-1} \ast \alpha \ast \lambda \). Then everything behaves exactly as for right group actions: \( \alpha \cdot (\lambda \ast \mu) = (\alpha \cdot \lambda) \cdot \mu \) and so on. The one obvious difference is that \( \lambda \ast \mu \) is only defined when \( \lambda \) ends where \( \mu \) begins. Note in particular that \( \pi_1(X, x_0) \) acts on itself by conjugation, but it is right conjugation.

By a right \( \Theta(X) \) action on a category \( \mathcal{C} \) (typically \( \mathcal{C} \) is \( \text{Set}, \text{Grp}, \text{Ab} \ldots \) we therefore mean a contravariant functor \( \Theta(X) \to \mathcal{C} \) and so haven’t defined anything new. It’s just a convenient way of denoting things. When \( \mathcal{C} \) is a category of modules (such as \( \text{Ab} \)), yet another term is local coefficient system of modules.
Now let \((Y, y_0)\) be a well-pointed space (so the inclusion \(\{y_0\} \to Y\) is a closed cofibration). The basepoint of \(Y\) is fixed throughout. For each choice of basepoint \(x_0 \in X\), let \(S(x_0) = [(Y, y_0), (X, x_0)]_*\) (pointed homotopy classes of pointed maps). If \([\lambda]: x_0 \to x_1\) is a path-class and \([f]\) \in S(x_0), define \([f] \cdot [\lambda] \in S(x_1)\) as follows:

- Choose a path \(\lambda\) representing the path class.
- Choose a map \(f\) representing the homotopy class.
- Choose a homotopy \(H: Y \times I \to X\) with \(H_0 = f\) and \(H(y_0, t) = \lambda\) (\(H\) exists by the cofibration assumption).
- Set \([f] \cdot [\lambda] = [H_1]\).

It is an elementary but rather lengthy exercise to show that the end result is independent of the three choices. You’ll find yourself using the homotopy extension property to construct homotopies of homotopies. Once this is done, it is an easier exercise to prove:

**Proposition 1.1**  

- The above construction gives a right action of \(\Theta(X)\) on sets.
- If \(Y = S^1 \wedge Z\), so that \(S(x_0)\) is a group, then we have an action on groups (i.e. right multiplication by \(\lambda\) is a group isomorphism).

**Remark.** Suppose \(X\) is path-connected, and fix a basepoint \(x_0\). Suppose we are given a right action of \(\Theta(X)\) in a category \(\mathcal{C}\) (e.g. \(\textbf{Set}, \textbf{Grp}\)\ldots\), i.e. a contravariant functor \(F: \Theta(X) \to \mathcal{C}\). Then \(\pi_1(X, x_0)\) acts on the right of \(F(x_0)\), and our functor is uniquely determined by this action. This reflects the fact that in a connected groupoid, i.e. a groupoid in which any two objects are isomorphic, the inclusion of the full subcategory defined by any one object is an equivalence of categories.

In particular we get a right action on groups by taking \(Y = S^n\) for \(n \geq 1\). The case \(n = 1\) is just the usual action on fundamental groups. For \(n > 1\) we get a local coefficient system \(x_0 \mapsto \pi_n(X, x_0)\). Before discussing the latter, however, we use the \(\Theta(X)\) action to clarify the relationship between pointed and unpointed (or “free”) homotopy classes. Note that for a fixed basepoint \(x_0 \in X\), we have a right action of \(\pi_1(X, x_0)\) on \([Y, X]_*\).

**Proposition 1.2** Suppose \(X\) is path-connected, and let \(x_0\) be a fixed basepoint. Then the natural map \([Y, X]_* \to [Y, X]\) from pointed to free homotopy classes induces a bijection \([Y, X]_* / \pi_1(X, x_0) \xrightarrow{\cong} [Y, X]\).

**Proof:** Suppose \(f: Y \to X\) is based at \(x_1\). Choose a path \(\lambda: x_1 \to x_0\). Then by definition of the action, \(f \cdot \lambda\) is based at \(x_0\) and freely homotopic to \(f\). This yields the surjectivity. If \(f\) is based at \(x_0\) and \(\lambda\) is a loop based at \(x_0\), then again by definition, \(f \cdot \lambda\) is freely homotopic to \(f\). So we get a factorization through the \(\pi_1(X, x_0)\)-orbits as shown. Finally, suppose \(f, g\) are based at \(x_0\) and freely homotopic. Let \(H\) be a free homotopy between them, and set \(\lambda = H(x_0, t)\). Then \(\lambda\) is a loop at \(x_0\) and \(f \cdot \lambda = g\). This yields the injectivity.
Corollary 1.3 If $X$ is simply-connected, $[Y, X]_* = [Y, X]$.

Here’s another situation in which pointed and free homotopy classes agree:

Proposition 1.4 Suppose $X$ is a path-connected $H$-space with $x_0$ a strict 2-sided identity (recall this can always be arranged if $(X, x_0)$ is well-pointed). Then $[Y, X]_* = [Y, X]$.

Proof: Let $m : X \times X \to X$ be the multiplication. Given a pointed map $f : Y \to X$ and a loop $\lambda$ at $x_0$, define a homotopy $H : Y \times I \to X$ by $H(y, t) = m(f(y), \lambda(t))$. Then $H_0 = f$ and $H(y_0, t) = \lambda(t)$, so by definition $H_1 = f \cdot \lambda$. But $H_1 = f$, so $\pi_1(X, x_0)$ acts trivially on $[X, Y]_*$ as desired.

Example. Any $K(G, 1)$ with $G$ abelian is an $H$-space, so pointed and free homotopy classes of maps into it coincide. If $G$ is non-abelian this is never true, since $\pi_1 K(G, 1) = G$ whereas $[S^1, K(G, 1)]$ is the set of conjugacy classes of $G$.

We now turn to the action of $\pi_1$ on higher homotopy groups. To get examples where this action is non-trivial, assume $X$ has a universal cover $p : \tilde{X} \to X$, and suppose $X$ is $(n - 1)$-connected (note this is automatic if $n = 2$). Using the Hurewicz theorem we have $\pi_n X \cong \pi_n \tilde{X} \cong H_n \tilde{X}$.

Exercise. Under these isomorphisms, the action of $\pi_1 X$ on $\pi_n X$ corresponds to the deck-transformation action on $H_n \tilde{X}$.

Now we can give some examples:

Example. Let $X = \mathbb{R}P^n$ with $n$ even. Then $\pi_n X \cong \mathbb{Z}$, and $\pi_1 X$ acts by $\lambda \mapsto -\lambda$.

Example. Let $X = S^1 \vee S^n$, $n \geq 2$. Then $\tilde{X}$ is an infinite string of balloons, i.e. the real line with copies of $S^n$ glued at one pole to the integer points. Note $\tilde{X}$ is $(n - 1)$-connected. So $\pi_n X = \bigoplus_{-\infty}^{\infty} \mathbb{Z}$, with $\pi_1 X \cong \mathbb{Z}$ acting by the obvious shift; in other words, as a module over the group ring $\mathbb{Z}[\pi_1 X]$, a.k.a. the Laurent polynomials, $\pi_n X$ is free on one generator.

Remark. Note this last example shows that the homotopy groups of a finite complex need not be finitely-generated. In the example we have that $\pi_n \tilde{X}$ is finitely-generated as a module over $\mathbb{Z}[\pi_1 X]$, but even this fails in general; see Hatcher p. 423. It was shown by Serre, using his spectral sequence, that the homotopy groups of a simply-connected finite complex are finitely-generated.

2 Further remarks

I’ll conclude with some general remarks.

1. Getting beyond simple-connectivity hypotheses. As the above examples show, the fundamental group can wreak havoc on the higher homotopy groups. Consequently many theorems seem to require that the spaces in question be simply-connected, a rather restrictive hypothesis for many purposes. For example, Serre’s finite-generation theorem mentioned
above, and the homology version of Whitehead’s theorem, are often stated only for simply-connected spaces. However, often one can get by with knowing that the action of \( \pi_1 \) on the higher homotopy groups is trivial (e.g. \( H \)-spaces). Spaces with this property are traditionally called simple spaces, although this is a horrible use of an already overworked word. It clashes badly with “simple modules”, for instance. It is also an unwieldy property to use, because the class of trivial modules over a group ring isn’t closed under extensions. This leads to the definition of a nilpotent space, meaning a space whose fundamental group is nilpotent and acts nilpotently on the higher homotopy groups; i.e. \( \pi_n X \) has a finite filtration with trivial action on the quotients. The class of nilpotent spaces is essentially the end of the road; one can generalize e.g. Whitehead’s theorem this far but (as far as I know) no further. Note that neither of the two examples above is nilpotent.

2. Locally constant sheaves. Suppose \( X \) is locally simply-connected. Then a local coefficient system of abelian groups on \( X \) is the same thing as a locally constant sheaf. To see this, suppose we are given a local coefficient system of abelian groups. This means that each \( x \in X \) we have an abelian group \( A_x \), together with a right action of the fundamental groupoid. On any simply-connected neighborhood \( U \) of some \( x_0 \) we get a trivialization \( \coprod_{x \in U} A_x \cong U \times A_{x_0} \) by choosing arbitrary paths in \( U \) from each \( x \) to \( x_0 \). This yields a topology on \( E := \coprod_{x \in X} A_x \) making \( E \rightarrow X \) a locally trivial discrete abelian group bundle. The local sections of this bundle then define a locally constant sheaf. Conversely any locally constant sheaf gives rise to a local coefficient system where the groups \( A_x \) are the stalks of the sheaf.

3. The local coefficient system of a fibration We’ll discuss this in detail when we get to spectral sequences. For now I’ll just briefly indicate the idea. One of the most common situations in which local coefficients systems arise is in the context of fiber bundles (think of the sphere bundle associated to a vector bundle with Riemannian metric, for instance). For example, we can consider the Klein bottle as an \( S^1 \)-bundle over the base space \( S^1 \). If we take the fiber over a basepoint and translate it around the base space circle, by the time the fiber \( S^1 \) returns to the basepoint it has been reflected by \( z \mapsto \bar{z} \). This transformation is sometimes called “monodromy”. It has degree \(-1\) so in homology we get the sign representation of the fundamental group. More generally for any fiber bundle with path-connected base \( B \) and fiber \( F \), this construction yields a representation of \( \pi_1 B \) on \( H_* F \), hence a local coefficient system. One can define homology groups with coefficients in a local coefficient system such as \( H_* F \), and these groups form the initial term of the Serre spectral sequence converging to the homology of the total space.