Exercises on characteristic classes

April 24, 2016

1. a) Compute the Stiefel-Whitney classes of the tangent bundle of $\mathbb{R}P^n$. (Use the method from class for the tangent Chern classes of complex projective spaces.)
   b) Conclude that if the tangent bundle is trivial, then $n = 2^m - 1$ for some $m$. (In fact $n$ must be 0, 1, 3, 7, but this is much harder to prove; one proof uses the Bott periodicity theorem.)
   c) Deduce (very easily!) a complete characterization of which real and complex projective spaces admit a spin structure. Recall that having a spin structure is equivalent to $w_1(\tau) = w_2(\tau) = 0$, where $\tau$ is the tangent bundle.

2. Let $\lambda, \xi$ be respectively a complex line bundle and a complex vector bundle of dimension $n$ over $B$. Find a formula for $c(\lambda \otimes \xi)$ in terms of $c(\lambda), c(\xi)$.

3. Suppose the tangent bundle of $S^n$ admits a complex structure. Show that $n = 2 \mod 4$. (In fact $n = 2, 6$, but the proof of the stronger statement uses the Bott periodicity theorem. See below under “Chern character” for more information.)

4. a) Let $Q$ denote the quaternion group of order 8, and let $V$ denote its unique irreducible real representation of dimension 4 (i.e., $\mathbb{H}$). Show that $w_4(V) \in H^*BQ$ is non-nilpotent, and all other $w_i = 0$ ($i > 0$).
   b) Let $D$ denote the dihedral group of order 8, and let $V$ denote its unique irreducible real representation of dimension 2. Show that $w_1 V, w_2 V$ are algebraically independent in $H^*BD$.
   Recall here that characteristic classes of a representation $V$ of $G$ are by definition the characteristic classes of the vector bundle $EG \times_G V \to BG$.

5. Additive characteristic classes. A characteristic class $f \in H^{2k}BU(n)$ is called additive or primitive if $f(\xi \oplus \eta) = f(\xi) + f(\eta)$ for all vector bundles $\xi, \eta$.
   a) Show that there is a unique additive class $s_k$ such that for line bundles $\lambda$, $s_k(\lambda) = c_1(\lambda)^k$, and that $s_k$ generates the group of all additive classes in $H^{2k}BU(n)$.
   b) Note that expressing $s_k$ as a polynomial in the Chern classes is equivalent to expressing $\sum y_i^k$ as a polynomial in the elementary symmetric functions. This leads to Newton’s recursion formula

$$s_n = c_1s_{n-1} + \ldots + (-1)^nnc_n = 0.$$
Prove this either from the symmetric function viewpoint, e.g. using the hint in the first part of Milnor's problem 16A, or by jumping ahead to the Hopf algebra viewpoint in a problem below.

c) As a sample calculation, compute the $s_k$'s for the tangent bundle of $\mathbb{C}P^n$. A more interesting example can be found in the next problem on hypersurfaces.

6. The topology of nonsingular complex hypersurfaces. I'll use the following notation:

- $i : V \subset \mathbb{C}P^{n+1}$ is a nonsingular complex hypersurface, of degree $d$ (so it is the zero set of a homogeneous polynomial of degree $d$).
- Let $y \in H^2\mathbb{C}P^{n+1}$ denote the generator characterized by $\langle y, [\mathbb{C}P^1] \rangle = 1$, i.e. $y = c_1 \lambda^*$.
- $z = i^* y$.
- $\tau_V, \nu_V \downarrow V$ are respectively the tangent bundle of $V$ and the normal bundle in $\mathbb{C}P^{n+1}$.

a) Let $u \in H^2\mathbb{C}P^{n+1}$ be Poincaré dual to $[V]$. Then $u = dy$. (The approach I have in mind involves transversality.)
b) $\langle z^n, [V] \rangle = d$.
c) $c(\tau_V) = (1 + z)^{n+2}/(1 + dz)$.
d) $\chi(V) = n + 2 + \frac{1}{d}((1-d)^{n+2} - 1)$. (Use the Euler class, (c), and some algebraic manipulation.)
e) Using the Lefschetz theorem on hyperplane sections (cf. Milnor's Morse theory) one can show that $i_*$ is an isomorphism on $H_k$ for $k < n$ and an epimorphism for $k = n$. Use this fact and part (d) to compute $H_* V$, showing in particular that the middle dimensional homology $H_n$ is free abelian with

$$\text{rank } H_n V = \begin{cases} 
\chi - n & \text{if } n \text{ even} \\
n + 1 - \chi & \text{if } n \text{ odd}
\end{cases}$$

Aside: The following special cases are worth noting:

1. In the quadric case $d = 2$, $\text{rank } H_n$ is zero for $n$ odd and 2 for $n$ even. The quadrics are special, because it turns out they are homogeneous spaces and even flag varieties of $SO(n + 1, \mathbb{C})$. The homology calculation can then be seen using Schubert cells.

2. The case $n = 2, d = 4$ is the K3 surface ("surface" in the complex sense), much studied in algebraic geometry and 4-manifold theory. Here can say "the" K3 surface because we are only talking about smooth hypersurfaces up to diffeomorphism; see the remark on Thom's theorem below. Note $\text{rank } H_n = 22$ in this case.

f) Let $s_n$ be the primitive characteristic class considered in exercise (5). Then

$$\langle s_n(\tau_V), [V] \rangle = d(n + 2 - d^n).$$

Remarks: 1. A theorem of Thom (with an interesting, accessible proof) says that for fixed $n, d$, any two such hypersurfaces are diffeomorphic, and are even ambient isotopic in $\mathbb{C}P^{n+1}$. So for example they are all diffeomorphic to the Fermat hypersurface $\sum z_i^d = 0$. 

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2. Part (f) has an important consequence for complex cobordism, in the case \( d = p \) a prime and \( n + 1 \) a power of \( p \). Note that in that case the value obtained is \( p \) times a \( p \)-local unit. This means that these particular \( V \)’s, which we can take to be the Fermat hypersurfaces of degree \( p \) in \( \mathbb{CP}^n \), are polynomial generators for a \( p \)-local summand of the complex cobordism ring called Brown-Peterson homology. I’ll explain all this further in class.

7. Grothendieck’s construction of Chern classes. Let \( p : E \rightarrow B \) be a complex vector bundle of dimension \( n \), and let \( \pi : P(E) \rightarrow B \) denote the associated projective space bundle. Then there is a canonical line bundle \( \lambda \) over \( P(E) \), where the fiber of \( \lambda \) over a line \( L \subset E_b \) is just \( L \). Let \( x = c_1(\lambda) \in H^2P(E) \). Regarding \( H^*P(E) \) as an algebra over \( H^*B \), show that

\[
x^n + c_1(E)x^{n-1} + c_2(E)x^{n-2} + \ldots + c_{n-1}(E)x + c_n(E) = 0.
\]

*Hint:* Look for a suitable bundle over \( P(E) \) that has a non-vanishing section and hence zero Euler class. The Euler class is the top Chern class; computing it will lead to the above formula.

*Remark:* Using the Serre spectral sequence or the Leray-Hirsch theorem (cf. Hatcher for the latter) one can show that \( H^*P(E) \) is a free module over \( H^*B \) on \( 1, x, \ldots, x^{n-1} \). It follows that there are unique elements \( a_1, \ldots, a_n \in H^*B \) such that \( x^n + a_1x^{n-1} + \ldots + a_n = 0 \). Therefore one could define \( c_k(E) = a_k \); this is Grothendieck’s definition (which you have just proved agrees with ours).

8. The Hopf algebra viewpoint. The Whitney sum formula gives \( H^*BU \) the structure of a bicommutative Hopf algebra (bicommutative means both the multiplication and the comultiplication are commutative). To explain this, let \( \mathcal{R} = \mathbb{Z}[c_1, c_2, \ldots] \) be the polynomial algebra on all the Chern classes. In other words, \( \mathcal{R} = H^*BU \), but I want to simultaneously present a purely algebraic approach that is also useful. I claim that \( \mathcal{R} \) is a Hopf algebra with diagonal \( \mu : \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R} \) given by

\[
\mu(c_k) = \sum_{i+j=k} c_i \otimes c_j.
\]

We can prove this claim in two ways:

a) Algebraic: Show by direct computation that \( \mu \) is coassociative and cocommutative (this is trivial once you unwind the definitions).

b) Topological: The space \( BU \) has the structure of a (homotopy) commutative Hopf group, with the multiplication \( m : BU \times BU \rightarrow BU \) coming from Whitney sum of vector bundles. To be precise, \( m \) is the unique map, up to homotopy, such that for all \( m, n \) the following diagram is commutative:

\[
\begin{array}{ccc}
BU(m) \times BU(n) & \longrightarrow & BU(m + n) \\
\downarrow & & \downarrow \\
BU \times BU & \underset{m}{\longrightarrow} & BU
\end{array}
\]
Here the top arrow is induced directly by Whitney sum, and the vertical arrows are the obvious ones. Although this is all seems intuitively plausible, there is a subtle point involved in proving the uniqueness. Let’s assume this fact for now; I’ll discuss the proof later. Then $H^*BU$ is a bicommutative Hopf algebra with $\mu = m^*$. The exercise is to check that $\mu$ is given by the formula above (this follows easily from the Whitney sum formula).

c) Show that additive characteristic classes correspond to primitive elements of the Hopf algebra $H^*BU$. Then prove Newton’s recursion formula by the following “generating function” approach: Let $c(t)$ denote the formal power series $\sum c_n t^n$. Then the “logarithmic derivative” $c'/c$ is defined; define $\bar{s}_n$ by $tc'/c = \sum_{n=1}^\infty \bar{s}_n t^n$. Now show:

1. $\bar{s}_n$ is primitive (use the fact that the logarithmic derivative converts multiplication to addition).

2. $\bar{s}_n = (-1)^{n+1} s_n$ (note you only need to check the coefficient of $c_1^n$ on both sides).

3. $\bar{s}_n + c_1 \bar{s}_{n-1} + \ldots + c_{n-1} \bar{s}_1 = nc_n$.

Newton’s formula now follows from c3.

d) Note that $H^*BU$ is the $\mathbb{Z}$-dual Hopf algebra to $H^*BU$ (or just take the dual of $\mathcal{R}$). Work out the details of the following: Let $j : \mathbb{C}P^\infty \to BU$ denote the map classifying the dual of the canonical line bundle $\lambda$ (the choice of the dual is a convention I’ll explain shortly). Let $b_n = j_\ast [\mathbb{C}P^n]$. Then $H_*BU$ is a polynomial algebra on the $b_i$’s, and moreover the comultiplication is given by $\Delta_* b_n = \sum_{i+j=n} b_i \otimes b_j$. In other words, our Hopf algebra is “self-dual”; $H_*BU$ is isomorphic to $H^*BU$ as a Hopf algebra.

If we use the monomial basis in the $c_i$’s for $H^*BU$, $b_n$ is dual to $c_1^n$. If one wants to work purely algebraically, this could be taken as the definition of $b_n$. Also, this is the reason I prefer to use $\lambda^*$ above; $\lambda$ would work too but then $b_n$ is dual to $(-1)^n c_1^n$ in the monomial basis. The self-duality ensures that likewise $c_n$ is dual to $b_1^n$ in the monomial basis in the $b_i$’s for $H_*BU$, but it’s also instructive to look at this topologically (i.e. think about the Chern classes of the external sum of $\lambda$’s over a product of $\mathbb{C}P^1$’s).

e) This exercise continues, but it’s getting too long! Assuming we get this far, we’ll pick it up later.

9. The Chern character and the Bott integrality theorem. In this problem all bundles are complex and cohomology has rational coefficients. Define the Chern character of a vector bundle $\xi$ of dimension $d$ by the formal sum

$$ch(\xi) = d + \sum_{n=1}^\infty \frac{s_n(\xi)}{n!}.$$ 

If the base space $X$ is finite-dimensional, this is an actual inhomogeneous element of $H^*(X; \mathbb{Q})$. In general it can be interpreted as a formal power series by inserting $t^n$’s, but notice that since the $n$-th term lies in $H^{2n}$, this is redundant; we can simply regard it as an element of the infinite product $\prod_{k=0}^\infty H^k X$ or equivalently as an element of the inverse limit of the truncated cohomology rings $H^{\leq k}$.

a) Note that $ch$ is additive in $\xi$, since this is true for the dimension $d$ and for the $s_n$’s. Show that it is also multiplicative with respect to tensor products:
\[ ch(\xi \otimes \eta) = ch(\xi)ch(\eta). \]

The multiplication on the right is as formal power series, although again the power series variable \( t \) can be omitted because the grading takes care of it.

b) The Bott Integrality theorem (a corollary of Bott periodicity) says that for any complex vector bundle \( \xi \) over \( S^n \), \( \langle ch(\xi), [S^n] \rangle \) is an integer. Assuming this, prove that if the tangent bundle of \( S^n \) has a complex structure, then \( n = 2 \) or \( n = 6 \). (Okay, \( n = 0 \) works too.)

10. **Thom’s theorem on Stiefel-Whitney classes.** This problem requires familiarity with Steenrod operations, although the axioms for the \( Sq^k \)'s are all you need to know. Let \( \xi \downarrow X \) be a real vector bundle over \( X \), of dimension \( n \). Let \( T(\xi) \) be the Thom space, \( u \) the Thom class. Show that \( Sq^k u = w_k(E)u \). (Define \( a_k \) by \( Sq^k u = a_k u \) and show that the \( a_k \)'s satisfy the axioms for the Stiefel-Whitney classes.)