The Jacobson radical

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At this point we have a good understanding of semisimple rings, in that we know they are all products of matrix rings over division algebras. Of course, the details of this decomposition for any particular such ring may be another matter entirely, but at least we have a good theoretical framework to work in. The next step is to allow rings that aren't semisimple, such as $\mathbb{F}_p G$ for a finite group whose order is divisible by p. The Jacobson radical is a useful tool for understanding the structure of such rings. For example, it is used to measure the failure of semisimplicity. Another application is to the striking theorem: Every left artinian ring is left noetherian.

1 Definitions

Without further ado, here is the definition:

Definition: The *Jacobson radical* $J(R)$ of a ring R is

$$J(R) = \cap_{W \text{ simple}} ann\, W.$$

In other words, $J(R)$ is the intersection of the annihilators of all simple R-modules. Note that $ann\, W$ is a 2-sided ideal, since it is the kernel of the natural homomorphism $R \longrightarrow End_{\mathbb{Z}} W$, and hence $J(R)$ is a 2-sided ideal. It is a proper ideal since $1 \notin J(R)$. Before giving examples we note the following equivalent formulation:

Proposition 1.1

$$J(R) = \cap_{\mathfrak{m}} \mathfrak{m},$$

where $\mathfrak{m}$ ranges over all maximal left ideals.

Proof: Suppose $x \in J(R)$ and $\mathfrak{m}$ is a maximal left ideal. Then $R/\mathfrak{m}$ is a simple module, so $x(R/\mathfrak{m}) = 0$ and hence $x \in \mathfrak{m}$. Conversely suppose x lies in every maximal left ideal, and W is a simple R-module. Then for every nonzero $w \in W$, $Rw = W$ and $ann\, w$ is a maximal left ideal. Hence $xw = 0$. So $xW = 0$, showing $x \in J(R)$.

A ring R is *local* if it has a unique maximal left ideal $\mathfrak{m}$.

Corollary 1.2 *If R is local with maximal left ideal $\mathfrak{m}$, then $J(R) = \mathfrak{m}$. In particular, $\mathfrak{m}$ is a 2-sided ideal.*

Example. We've seen a few examples of commutative local rings (and we'll see many more in the spring). For example $J(\mathbb{Z}_{(p)}) = (p)$, and if F is a field then $J(F[[x]]) = (x)$.

Example. Suppose F is a field of characteristic p and G is a finite p-group. Then FG is a local ring with maximal ideal the augmentation ideal IG, so $J(FG) = IG$. (Exercise.)

Note that if R is local with maximal ideal $\mathfrak{m}$, then $R/\mathfrak{m}$ is a division ring (because it has no proper nonzero left ideals). For us, it is almost invariably a field (even when R is noncommutative).

2 Semisimplicity and the Jacobson radical

We can see right away that $J(R)$ is related to semisimplicity:

Theorem 2.1 *If R is semisimple, then $J(R) = 0$. The converse holds if R is left artinian.*

Proof: We have seen that if R is semisimple, then as a left module it is a direct sum (indeed a finite direct sum) of simple modules. So $x \in J(R) \Rightarrow xR = 0 \Rightarrow x = 0$.

Now suppose R is left artinian and $J(R) = 0$. We will show that the left regular module R is a direct sum of simple modules, so R is semisimple. The first ingredient doesn't use the artinian condition:

Lemma 2.2 *Let R be any ring with $J(R) = 0$. Then every minimal left ideal is a direct summand (of the left regular module).*

Proof: Let I be a minimal left ideal. Since $J(R) = 0$, there is a maximal left ideal $\mathfrak{m}$ that doesn't contain I. Since I is a simple R-module, we then have $I \cap \mathfrak{m} = 0$. Since $\mathfrak{m}$ is maximal, $I + \mathfrak{m} = R$. So $I \oplus \mathfrak{m} = R$, proving the lemma.

We will also use the following trivial fact: Suppose M is an R-module, and $L \subset N \subset M$ are submodules such that L is a direct summand of M. Then L is a direct summand of N.

Now since R is left artinian, it has a minimal left ideal I_1. By the lemma I is a direct summand, say $R = I \oplus K_1$. Again by the artinian condition, K_1 contains a minimal left ideal I_2. By the lemma and the trivial fact, I_2 is a direct summand of K_1, say $K_1 = I_2 \oplus K_2$. Continuing in this way we obtain a descending chain of left ideals $K_1 \supset K_2 \supset ...$ that must stabilize since R is left artinian. But if $K_n = K_{n+1}$, we must have $K_n = 0$ (otherwise K_n would contain a minimal left ideal and the process would continue). So $R = I_1 \oplus ... \oplus I_n$. This completes the proof of the theorem.

The converse is false without the artinian hypothesis. For example, $J(\mathbb{Z}) = 0$ but $\mathbb{Z}$ is not semisimple. The condition $J(R) = 0$ is sometimes called "Jacobson semisimple".

Now consider the quotient homomorphism $\pi : R \longrightarrow R/J(R)$. Since $J(R)$ annihilates simple modules by definition, every simple R-module is pulled back from a simple $R/J(R)$-module, and in fact π^* defines a bijection between isomorphism classes of simple $R/J(R)$-modules and isomorphism classes of simple R-modules. In particular, we have at once:

Proposition 2.3 $J(R/J(R)) = 0$.

Thus $R/J(R)$ can be viewed as the maximal Jacobson semisimple quotient of R. This is most interesting, however, when R is left artinian. Then:

Corollary 2.4 *If R is left artinian, $R/J(R)$ is the maximal semisimple quotient of R.*

Combining the correspondence between simple R-modules and simple $R/J(R)$-modules with the Artin-Wedderburn theory, we have:

Theorem 2.5 *Suppose R is left artinian. Let $W_1, ..., W_m$ denote the distinct simple R-modules. Then*

$$R/J(R) \cong \prod_{i=1}^{m} M_{n_i} D_i,$$

where n_i is the multiplicity of W_i in $R/J(R)$ and $D_i^{op} \cong End_R W_i$.

See the exercises for examples.

3 Nakayama's lemma

Although Nakayama's lemma is rather obscure at first encounter, it is extraordinarily useful and easy to prove. Rest assured that the point of it will become clearer as we proceed.

Proposition 3.1 *If M is a finitely-generated R-module and $J(R)M = M$, then $M = 0$.*

Proof: Let $J = J(R)$, and suppose $M \neq 0$. Since M is finitely-generated it has a non-trivial cyclic quotient, hence a non-trivial simple quotient W. Let $\pi : M \longrightarrow W$ denote the quotient homomorphism. Then $JW = 0$ so $JM \subset Ker\,\pi$. Hence $JM \neq M$.

Note the special case when R is local with maximal left ideal $\mathfrak{m}$: If M is finitely-generated and $\mathfrak{m}M = M$ then $M = 0$. Note also that the finitely-generated hypothesis is essential. For example if R is the local ring $\mathbb{Z}_{(p)}$ and $M = \mathbb{Q}$, then $(p)M = M$ but $M \neq 0$.

One common use of Nakayama's lemma in the local case is as follows.

Proposition 3.2 *Let R be a local ring with maximal left ideal $\mathfrak{m}$, and let M be a finitely-generated R-module. Then a finite set of elements $x_1, ..., x_n \in M$ generates M if and only if their images $\overline{x}_1, ...\overline{x_n}$ in $M/\mathfrak{m}M$ generate $M/\mathfrak{m}M$.*

Proof: The "only if" is trivial. Now suppose the $\overline{x}_i$'s generate $M/\mathfrak{m}M$. Let $N \subset M$ denote the submodule generated by the x_i's. Since the restriction to N of the quotient homomorphism $M \longrightarrow M/\mathfrak{a}M$ is surjective by assumption, we have $M = \mathfrak{m}M + N$. It follows that $\mathfrak{m}(M/N) = M/N$, and hence $M/N = 0$ by Nakayama's lemma, i.e. $N = M$ as desired.

Corollary 3.3 *Let $D = R/\mathfrak{m}$, and recall D is a division algebra. Then the x_i's form a minimal generating set for M if and only if the $\overline{x}_i$'s form a D-basis for $M/\mathfrak{m}M$.*

Here we begin to see the utility of Nakayama's lemma, which indeed is one of the properties of local rings that makes them easier than arbitrary rings. For example, if we have a finitely-generated free $\mathbb{Z}_{(p)}$-module M, then any $\mathbb{F}_p$-basis for M/p lifts to a $\mathbb{Z}_{(p)}$-basis of M. On the other hand if N is a free abelian group and we choose a prime p, the analogous statement for N/p is false even in the rank 1 case.

4 Units and the Jacobson radical

There are a number of characterizations of $J(R)$ involving units, some of which are rather obscure. The most basic, and easiest to understand and use, is the following:

Proposition 4.1 *The following are equivalent:*
a) $x \in J(R)$*.*
b) For all $r \in R$*,* $1+rx$ *has a left inverse (i.e. there is an* $s \in R$ *such that* $s(1+rx) = 1$*).*
c) For all $r \in R$*,* $1 + rx$ *is a unit.*

Proof: (a) $\Rightarrow$ (b): Since $J(R)$ is a left ideal, it suffices to show $x \in J(R) \Rightarrow 1 + x$ has a left inverse. If not, then $R(1 + x)$ is a proper left ideal and hence there is a maximal left ideal $\mathfrak{m}$ with $1 + x \in \mathfrak{m}$. But $x \in \mathfrak{m}$, so $1 \in \mathfrak{m}$, a contradiction.

(b) $\Rightarrow$ (c): By assumption $1 + rx$ has left inverse, say $s(1 + rx) = 1$. Hence $s = 1 - srx$ also has a left inverse, again by assumption; say $ts = 1$. But if an element in a ring has both a left and right inverse, the two are equal. Here we conclude $t = 1+rx$. Hence s is a 2-sided inverse of $1 + rx$, i.e. $1 + rx$ is a unit.

(c) $\Rightarrow$ (a): Suppose $1 + rx$ is a unit for all r, and let $\mathfrak{m}$ be a maximal left ideal. If $x \notin \mathfrak{m}$ then $Rx + \mathfrak{m} = R$, so there is an $r \in R$ and $y \in \mathfrak{m}$ such that $rx + y = 1$. Then $y = 1 - rx$ is a unit by assumption, so $1 \in Ry$ and $Ry = R$, contradicting $y \in \mathfrak{m}$.

There is a more symmetric variant of the proposition, whose proof is left to the interested reader:

Proposition 4.2 $x \in J(R)$ *if and only if for all* $r, s \in R$*,* $1 + rxs$ *is a unit.*

Using this we can settle another symmetry issue which may have been bothering you:

Proposition 4.3 $J(R)$ *is the intersection of all maximal right ideals of* R*.*

Proof sketch. We defined $J(R)$ in terms of annihilators of simple left R-modules. Similarly we can define a 2-sided ideal $K(R)$ using simple right modules, and show that it is the intersection of all maximal right ideals of R. Then (not surprisingly in view of the symmetry), we can show that the previous proposition holds with K in place of J, and hence $K(R) = J(R)$.

5 Nilpotence and the Jacobson radical

If R is a commutative ring, then the set of nilpotent elements in R is an ideal (an easy exercise). This ideal is called the *nilradical* or just "the radical", and is denoted $\mathcal{N}(R)$ or $\mathcal{N}_R$. It plays an important role in commutative algebra. In the non-commutative case, however, the nilpotent elements need not be closed under addition, nor under multiplication by elements of R (for counterexamples, look in a 2×2 matrix ring over a field).

In noncommutative rings, the more relevant concept is often that of a *nilpotent ideal*: that is, a left ideal I such that $I^n = 0$ for some n. By definition, I^n consists of all sums of n-fold products of elements of I. In particular, every element $x \in I$ is nilpotent and in fact $x^n = 0$ for all x if $I^n = 0$.

Now if x is nilpotent, the *nilpotence order* of x is the minimal n such that $x^n = 0$. By definition, then, the elements of a nilpotent ideal have nilpotence order $\leq n$ for some fixed n. This makes it clear that even in a commutative ring, an ideal consisting of nilpotent elements need not be a nilpotent ideal. (For example, take a polynomial ring $F[x_1, x_2, ...]$ on infinitely many variable x_n and mod out the ideal generated by (x_n^n).)

Recall that if x is nilpotent, then $1 + x$ is a unit.

Proposition 5.1 *If I is a left ideal consisting of nilpotent elements (note this is weaker than I a nilpotent ideal), then $I \subset J(R)$.*

Proof: If $x \in I$ then rx is nilpotent for all $r \in R$, so $1 + rx$ is a unit for all $r \in R$. Hence $x \in J(R)$ by Proposition 4.1.

Corollary 5.2 *If R is commutative, then $\mathcal{N}(R) \subset J(R)$.*

Equality need not hold in the corollary. For example if $R = \mathbb{Z}_{(p)}$ then $\mathcal{N}(R) = 0$ and $J(R) = (p)$.

Proposition 5.3 *If R is left artinian, then $J(R)$ is nilpotent.*

Proof: The descending chain of ideals $J \supset J^2 \supset J^3 \supset ...$ must stabilize; say $J^n = J^{n+1}$ for $n \geq m$. (If J was finitely-generated as a left ideal we could apply Nakayama's lemma immediately to get $J^m = 0$, but we don't have this.) Let $I = J^m$. Then $I^2 = I$, and claim that $I = 0$. If not then the set of left ideals K such that $IK \neq 0$ is nonempty, and since R is left artinian there is a minimal such K. By minimality K is a principal left ideal, and in fact there is a $y \in K$ with $K = Iy$ (since Iy is a left ideal). Moroever $IK = I^2y = Iy = K$, so $JK = K$. Since K is generated by y and in particular is finitely-generated, Nakayama's lemma applies and we conclude $K = 0$, a contradiction.

There is a companion result for the nilradical in commutative rings:

Proposition 5.4 *Let R be a commutative ring. If R is noetherian, then $\mathcal{N}(R)$ is a nilpotent ideal.*

Proof: Since R is noetherian, every ideal in I is finitely-generated. In particular $\mathcal{N}(R)$ is finitely-generated, say by $x_1, ..., x_n$. Let d_i denote the nilpotence order of x_i, and let $d = d_1...d_n$. Then I claim $I^d = 0$. For I^d is spanned by length d products $y_1...y_d$ in which each y_j is one of the x_i's. By the definition of d, this means at least one x_i occurs at least d_i times, so $y_1...y_d = 0$.

We emphasize that this last result is for the nilradical, not the Jacobson radical. As we've seen, the Jacobson radical of a commutative noetherian ring need not be nilpotent (e.g. $\mathbb{Z}_{(p)}$).

6 On artinian versus noetherian rings

For modules over a ring R, the artinian and noetherian conditions are more or less on equal footing. A module M is artinian (resp. noetherian) if it satisfies the descending (resp. ascending) chain condition on submodules, and neither condition implies the other. But if the module in question is the left regular module R, a surprising asymmetry turns up.

Theorem 6.1 *If R is left artinian, then it is left noetherian.*

The converse is false (take $R = \mathbb{Z}$ for instance). Before proving the theorem, we recall two convenient facts (the proofs are straightforward).

Proposition 6.2 *Let M be a module over a ring R.*

a) M has finite length if and only if M is both artinian and noetherian.

b) If M is completely reducible, then the three properties artinian, noetherian, and finite length are equivalent.

Proof of theorem. We will show that the left regular module R has finite length (cf. fact (a)). Let $J = J(R)$. Since R is left artinian, J is nilpotent, say $J^n = 0$. Hence R has a finite decreasing filtration

$$R = J^0 \supset J \supset J^2 \supset ... \supset J^n = 0.$$

So it will suffice to show that each quotient $N_i := J^i/J^{i+1}$ of this filtration has finite length. Any subquotient of an artinian module is artinian, so N_i is an artinian R-module. But clearly N_i is pulled back from an R/J-module, and so is artinian as an R/J-module. Since R/J is semisimple (Corollary 2.4), N_i is completely reducible and hence has finite length by fact (b).

7 Exercises

1. Let $G = S_3$ (isomorphic to $GL_2\mathbb{F}_2$) and consider the group algebra $R = \mathbb{F}_pG$ for a prime p. Determine the simple R-modules, the Jacobson radical, and the maximal semisimple quotient. (The answer will depend on p, of course.)

While you're at it, prove the following: Suppose F is a field of characteristic p and G is a finite group with a normal p-Sylow subgroup, and consider the group algebra $R = FG$ for a prime p. Determine the Jacobson radical and the maximal semisimple quotient. (The case $p = 3$ of the S_3 problem is a special case.)

2. Same question as (1) for $\mathfrak{b}_n F$, the algebra of upper triangular matrices over a field F. (Determine the simple modules, Jacobson radical, maximal semisimple quotient.)

3. Let G be a finite p-group and let F be a field of characteristic p. Show that $J(FG) = IG$, where IG is the augmentation ideal.

4. A commutative ring is *semilocal* if it has only finitely many maximal ideals. Show that the Jacobson radical of a principal ideal domain R is nonzero if and only if R is semilocal and not a field.

Remark. One can easily construct examples of semilocal rings e.g. by imitating the construction of $\mathbb{Z}_{(p)}$. Let $p_1, ..., p_n$ be a finite set of primes, and let $R \subset \mathbb{Q}$ denote the subring of fractions a/b in lowest terms such that none of the p_i's divides b. Then R is a semilocal principal ideal domain with n maximal ideals, namely the ideals $p_i R$. Convince yourself that this is correct but don't write it up, as we'll have a much more systematic approach to such matters later (localization of commutative rings).