Injectives and Ext

This note finishes up Wednesday's lecture, together with the notes provided by Angus.

Suppose F is a left exact covariant functor from A-modules to B-modules. The right derived functors $F^n N$ of an A-module N are defined as follows:

1. Choose an injective resolution $N \longrightarrow I^{\cdot}$.
2. Apply the functor F to $I^{\cdot}$ (omit N) and take the cohomology of the resulting cochain complex: $F^n N := H^n(F(I^{\cdot}))$.

As before, this is independent of the choice of resolution, up to a canonical natural isomorphism, does indeed yield functors F^n, there is a long exact sequence associated to a short exact sequence of A-modules, etc. One of the main examples is to take a fixed A-module M and $F(N) = Hom_A(M, N)$. Here we can always take $B = \mathbb{Z}$. If A is commutative we can take $B = A$, and more generally if A is a K-algebra for some commutative ring K, we take $B = K$. We then have the "balance for Ext" theorem:

Theorem 0.1 *The derived functors $F^n(N)$ of $Hom_A(M, -)$ are naturally isomorphic to $Ext^n_A(M, N)$.*

In other words, you can compute the Ext using either a projective resolution of M or an injective resolution of N. The proof is a straightforward variant of the proof of balance for Tor; the latter will be given in the final lecture.

Examples. 1. Using a projective resolution of $\mathbb{Z}/n$, we showed that $Ext^i_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$ for $i = 1$ and is zero otherwise. We get the same result if we use the injective resolution of $\mathbb{Z}$ given by $\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$, as you can (trivially!) check.

2. Let K be a commutative ring, G a group and $A = KG$. The most basic functor associated to KG-modules is perhaps the invariants functor $M \longrightarrow M^G$, which is left exact. So we can form its derived functors using injective resolutions as above. Notice, however, that M^G is the same thing as $Hom_{KG}(K, M)$ where K has trivial G-action. Hence by the balance theorem the derived functors are the same as $Ext^*_{KG}(K, M)$, i.e. "group cohomology". In particular we can work with a fixed projective resolution of K, which is sometimes easier than injectively resolving each M separately.

Finally, just to pique your curiosity I'll mention one of the main examples of derived functors—going beyond our module categories—namely sheaf cohomology. I don't assume you know what a sheaf is exactly, but just play along anyway. For a sheaf $\mathcal{E}$ of modules on a topological space X there is a *global sections* functor $\mathcal{E} \mapsto \Gamma(\mathcal{E})$. Think of the sheaf of smooth differential forms on a smooth manifold for instance, or regard an A-module M as a sheaf on $Spec\, A$. In the former case a global section is a smooth p-form on X; in the latter it is just an element of M. This functor is left exact, and its derived functors are by definition the sheaf cohomology modules. For this to make sense one needs to first prove that the category of sheaves has "enough injectives", so that every sheaf has an injective resolution. This can be done. But it turns out that at this level of generality we don't have projective resolutions anymore, so there is not even the possibility of a "balance" theorem. Hence for sheaves, injective resolutions are all-important.