Notes on Homological Algebra

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June 4, 2015

Abstract

This is my notes on homological algebra. Due to insufficient time, this will only include elementary results on injective modules. Unless other specified, R refers to a ring with identity, not necessarily commutative. Any R-module is a left module, unless other specified.

Definition 1. An R-module I is **injective** if the functor $\mathrm{Hom}_R(-, I)$ is exact.

Since this functor is always left exact, it suffices to say that the functor is right exact in the definition. Hence, I is injective if

$$\begin{array}{ccc} M & \hookrightarrow & N \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \exists} & \\ I & & \end{array}$$

for all M, N and f. From this and universal property, it is easy to see that product of injective modules is injective. This is an analogy to the fact that direct sum of projective modules is projective.

Proposition 1. *I is injective if and only if every short exact sequence $0 \to I \to M \to N \to 0$ splits.*

Proof The forward direction is trivial. Suppose the latter statement is true, and we have the diagram after Definition 1 except the dashed map. Consider the commutative diagram

$$\begin{array}{ccc} M & \overset{i}{\hookrightarrow} & N \\ {\scriptstyle f}\downarrow & & \downarrow \\ I & \longrightarrow & (I \oplus N)/\langle f(x) - i(x) : x \in M \rangle \end{array}$$

where the bottom right is the pushout of f and i. The bottom map is injective, since $(f(x), 0) \in I \oplus N$ is in the ideal $\langle f(x) - i(x) \rangle$ if and only if $f(x) = 0$. By assumption, we have a map from $(I \oplus N)/\langle f(x) - i(x) : x \in M \rangle \to I$ such that $I \to (I \oplus N)/\langle f(x) - i(x) : x \in M \rangle \to I$ is the identity. Our induced map $N \to I$ would then be the composition $N \to (I \oplus N)/\langle f(x) - i(x) : x \in M \rangle \to I$. ■

Recall that free modules are projective. However, free modules are not necessarily injective. For instance, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module, since the exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ does not split for all $n > 1$. A more direct way to see this is that there is $\mathbb{Z}$-module map $\mathbb{Q} \to \mathbb{Z}$ that makes the following diagram commute, since the only group homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$ is 0.

$$\begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathbb{Q} \\ {\scriptstyle Id}\downarrow & & \\ \mathbb{Z} & & \end{array}$$

A useful result on injective modules is Baer's Lemma.

Theorem 2 (Baer's Lemma)**.** *I is injective if and only if for all ideal $\mathfrak{a} \subset R$ and $f : \mathfrak{a} \to I$,*

$$\begin{array}{ccc} \mathfrak{a} & \hookrightarrow & A \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \exists} & \\ I & & \end{array}$$

Proof The forward direction is by definition. Suppose the latter statement is true, and suppose we have the diagram after Definition 1 except the dashed map. If N is cyclic, then $N \cong R/\mathfrak{b}$ for some ideal $\mathfrak{b} \subset R$. M is then an ideal in $R/\mathfrak{b}$, so there is an ideal $\mathfrak{a} \subset R$ such that $\mathfrak{a} \supset \mathfrak{b}$ and $M \cong \mathfrak{a}/\mathfrak{b}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{a} & \hookrightarrow & R \\ \downarrow & & \downarrow \\ \mathfrak{a}/\mathfrak{b} & \hookrightarrow & R/\mathfrak{b} \\ {\scriptstyle f}\downarrow & & \\ I & \xleftarrow{\exists \tilde{f}} & (R) \end{array}$$

The map $\tilde{f} : R \to I$ exists by assumption. Since $\mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{b}$ is annihilated in $\mathfrak{a} \to \mathfrak{a}/\mathfrak{b}$, $\tilde{f}(\mathfrak{b}) = 0$. Therefore $\tilde{f}$ factors through a map $f' : R/\mathfrak{b} \to I$, proving the claim for cyclic N.
We will complete the proof by Zorn's lemma. The collection $\{(M', f')\}$, where for each pair we have the following commutative diagram

$$\begin{array}{ccccc} M & \hookrightarrow & M' & \hookrightarrow & N \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & & & \\ I & & & & \end{array}$$

This is a partially ordered set, where the order is defined in the obvious way. It satisfies the criteria for Zorn's lemma. Hence there is a maximal $(N', \tilde{f})$. We claim that $N' = N$. If not, then there is $x \in N - N'$. Consider the following diagram

$$\begin{array}{ccc} N' \cap Rx & \hookrightarrow & Rx \\ {\scriptstyle \tilde{f}}\downarrow & \swarrow {\scriptstyle \exists} & \\ I & & \end{array}$$

The map from Rx to I exists since Rx is cyclic. Now we have maps $N' \to I$ and $Rx \to I$ that agrees on $N' \cap Rx$, so there exists a map $g : N' + Rx \to I$ such that the following diagram commutes.

$$\begin{array}{ccccccc} M & \hookrightarrow & N' & \hookrightarrow & N' + Rx & \hookrightarrow & N \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \tilde{f}} & & \swarrow {\scriptstyle g} & & & \\ I & & & & & & \end{array}$$

Since $N' \subsetneq N' + Rx$, this contradicts that N' is maximal from the collection. Therefore $N' = N$, completing the proof. ■

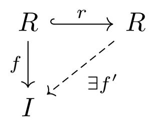
Next, we want to prove that every modules embeds in some injective module. To begin with, we need the notion of divisible modules.

Definition 2. Suppose R is a PID. An R-module M is **divisible** if for all $x \in M$ and for all $r \neq 0$ in R, there is $y \in M$ such that $x = ry$. In other words, the map $M \xrightarrow{r} M$ is surjective for all $r \neq 0$.

For example, if $R = \mathbb{Z}$, then $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}/p^\infty$ (the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the powers of $1/p$) are divisible $\mathbb{Z}$-modules. In particular, $\mathbb{Z}/p^\infty$ is divisible since we can easily invert the powers of p, and if $\gcd(r, p) = 1$ then r is a unit in $\mathbb{Z}/p^n\mathbb{Z}$ for all $n > 0$.
Given that R is an integral domain, there is a relationship between injective modules and divisible modules. In fact the two notions are equivalent if R is a PID.

Proposition 3. *Suppose R is an integral domain. If I is injective, then I is divisible. The converse is true if R is a PID.*

PROOF Suppose I is injective and let $x \in I$, $r \neq 0$ in R. Define an R-module map $f : R \to I$ by $f(s) = sx$. Consider the following commutative diagram.

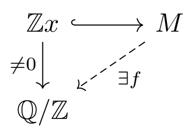


The map $R \xrightarrow{r} R$ is injective since $r \neq 0$ and R is an integral domain. Let $y = f'(1)$. Then $y = rx$ by commutativity of the diagram, showing that I is divisible.
Conversely, suppose R is a PID and I is divisible. By Baer's Lemma (2), it suffices to check the diagram for $\mathfrak{a} \hookrightarrow R$. Since R is a PID, all ideals in R are principal. Let (r) be an ideal in R. If $r = 0$ we can take the zero map from R to I. Now suppose $r \neq 0$. Since I is divisible, there is $y \in I$ such that $f(r) = ry$. Define $f' : R \to I$ by $f'(1) = y$. Then f' is our desired map, so I is injective. ■

As a corollary, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. We will need it for the following proposition.

Proposition 4. *Every $\mathbb{Z}$-module embeds in an injective $\mathbb{Z}$-module.*

PROOF We first claim that if M is a $\mathbb{Z}$-module and $x \neq 0$ is in M, then there is $f : M \to \mathbb{Q}/\mathbb{Z}$ with $f(x) \neq 0$. This is true if M is cyclic: suppose M is generated by y and $x = ny$ for some $n \neq 0$. If M is free then we can define $f(y) = 1/2n + \mathbb{Z}$. If M is a torsion module suppose $m > 0$ is the smallest positive integer that annihilates y. Since $x \neq 0$ we have $m \nmid n$. We can define $f(y) = 1/m + \mathbb{Z}$. In any case, the statement is true if M is cyclic. Now, the statement is true for any M and any $x \neq 0$ in M, by considering the following commutative diagram, using the fact that $\mathbb{Q}/\mathbb{Z}$ is injective.



Now we will prove the proposition. Construct the map

$$M \longrightarrow \prod_{\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$$

by

$$x \mapsto (f(x))_{f \in \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})}.$$

This map is injective by the claim. ■

Recall that we want to prove that every R-module embeds into an injective R-module. To do so, we still need a few steps.

Proposition 5. *If E is an injective $\mathbb{Z}$-module, then $\mathrm{Hom}_{\mathbb{Z}}(R, E)$ is an injective R-module, where the module structure is given by $(r \cdot f)(s) = f(sr)$.*

We need two lemmas to prove this.

Lemma 5.1. *The functor $\mathbb{Z}$-mod $\to$ R-mod by $E \mapsto \mathrm{Hom}_{\mathbb{Z}}(R, E)$ is right adjoint to the forgetful functor R-mod $\to$ $\mathbb{Z}$-mod.*

Proof Let $F : R\text{-mod} \to \mathbb{Z}\text{-mod}$ be the forgetful functor, and $G : \mathbb{Z}\text{-mod} \to R\text{-mod}$ be our given functor. We want to show that for all R-module M and $\mathbb{Z}$-module N, $\mathrm{Hom}_{\mathbb{Z}}(F(M), N)$ is naturally isomorphic to $\mathrm{Hom}_R(M, G(N))$. We define the forward map by sending $f : F(M) \to N$ to $\tilde{f}$, where $\tilde{f}(x) \in \mathrm{Hom}_{\mathbb{Z}}(R, E)$ is defined by $\tilde{f}(x)(r) = f(F(rx))$. The backward map is defined by sending $g : M \to G(N)$ to $\tilde{g}$, where $\tilde{g}(x) = g(x)(1)$. It is easy to check that the maps are well-defined, natural, and inverse to each other. ∎

Lemma 5.2. *Any covariant functor between module categories with an exact left adjoint preserves injective modules.*

Proof Suppose G is such a functor, right adjoing to F that is exact. Let I be an injective module in the first category, and M, N be modules in the second category such that we have the maps

$$\begin{array}{ccc} M & \hookrightarrow & N \\ {\scriptstyle f}\downarrow & & \\ G(I) & & \end{array}$$

Apply the functor F the the diagram. Since F is exact, $F(M) \hookrightarrow F(N)$. Since F is left adjoint to G, we have

$$\begin{array}{ccc} F(M) & \hookrightarrow & F(N) \\ {\scriptstyle F(f)}\downarrow & \swarrow {\scriptstyle \exists f'} & \\ I & & \end{array}$$

The map f' exists since I is injective. Apply the functor G and use the fact that G is right adjoint to F, we can conclude that $G(f') : N \to G(I)$ commutes the first diagram. Therefore, $G(I)$ is injective. ∎

We can now prove Proposition 5 easily. The fact that $\mathrm{Hom}_{\mathbb{Z}}(R, E)$ is an R-module under the given structure is easy to check. By applying the two lemmas above the proposition follows clearly. ∎

We can now prove our main goal.

Theorem 6. *Every R-module M embeds in some injective R-module. In particular, the map*

$$M \longrightarrow \prod_{\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$$

defined by

$$x \mapsto (\varphi(x))_{\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))}$$

is injective.

Proof It suffices to show that for every nonzero $x \in M$, there is a $\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$ such that $\varphi(x)$ is not the trivial map from R to $\mathbb{Q}/\mathbb{Z}$. By the proof of Proposition 4, there is $f \in Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ such that $f(x) \neq 0$. Define an R-module map $\varphi' : Rx \to Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ by $\varphi'(rx)(s) = f(srx)$. This is in fact an R-module map, since $(r \cdot \varphi'(y))(s) = \varphi'(y)(sr) = f(sry) = \varphi'(ry)(s)$. Since $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module, there is a map $\varphi : M \to Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ such that the following diagram commutes.

$$\begin{array}{ccc} Rx & \hookrightarrow & M \\ {\scriptstyle \varphi'}\downarrow & \swarrow {\scriptstyle \exists \varphi} & \\ \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) & & \end{array}$$

$\varphi(x)$ is not the trivial map from R to $\mathbb{Q}/\mathbb{Z}$, since $\varphi(x)(1) = \varphi'(x)(1) = f(x) \neq 0$. Hence the map we have in the theorem is injective. By Proposition 5, $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is injective. Since product of injective modules are injective, M embeds into some injective R-module, proving our theorem. ■

With this, we can easily construct an injective resolution for every R-module M.

Definition 3. Suppose M is an R-module. An **injective resolution** for M is a cochain complex of R-modules $I^\bullet$ such that every I^n is injective (a.k.a. **injective**), and the sequence $0 \to M \to I^0 \to I^1 \to \dots$ is exact (a.k.a. **acyclic**).

Theorem 7. *Every R-module M admits an injective resolution.*

PROOF Let M be an R-module. By Theorem 6, there is an injective R-module I^0 such that $M \hookrightarrow I^0$. We will then define I^n inductively for $n > 0$. Let N be the cokernel for the map $I^{n-2} \to I^{n-1}$ (for convenience let $I^{-1} = M$ here). By Theorem 6 again, there is an injective R-module I^n such that $N \hookrightarrow I^n$. Define the map $I^{n-1} \to I^n$ by composing the projection $I^{n-1} \to N$ and the embedding $N \to I^n$. The kernel of $I^{n-1} \to I^n$ is the same as the kernel of $I^{n-1} \to N$, which is the image of $I^{n-2} \to I^{n-1}$. Therefore the sequence $0 \to M \to I^\bullet$ is exact. ■