We continue to focus on representations of finite groups over $\mathbb{C}$. Although we have proved many interesting theorems on the number of irreducible representations and their dimensions, what’s missing is that we have very few ways to actually construct the irreducibles. The main topic of Part IV, *induced representations*, will provide a powerful new tool in this direction. For this purpose, we first introduce tensor products over an $F$-algebra. We also take the opportunity to discuss more systematically “pulling back” a representation of $G$ along a homomorphism $G' \to G$, and the resulting right action of $\text{Aut} G$ on $\text{Rep}_F G$. Among other things, this helps to clarify Mackey’s double-coset formula and irreducibility criterion.

Some of the general constructions below work over an arbitrary field and with any group. So we let $F$ denote any field, $G$ any group, until otherwise specified. We also point out that these constructions have simpler prototypes in the world of $G$-sets; you may find it helpful to read (or re-read) the $G$-set notes on “balanced products” and “induced $G$-sets”; see also the exercises to those notes.

1 Tensor products over $F$-algebras

Let $R$ be an $F$-algebra, and suppose given a right $R$-module $M$ and a left $R$-module $N$. Then the tensor product over $R$, denoted $M \otimes_R N$, is the $F$-module defined by $M \otimes_R N = (M \otimes_F N)/Z$, where $Z$ is the $F$-submodule generated by all elements of the form $xr \otimes y - x \otimes ry$ ($x \in M$, $y \in N$, $r \in R$). Note that in general there is no natural $R$-module structure on $M \otimes_R N$; it is only an $F$-module. If $R$ is commutative, however, then there is no right/left distinction, and moreover $M \otimes_R N$ has a natural $R$-module structure given by $r \cdot (x \otimes y) = (rx) \otimes y = x \otimes (ry)$. This is analogous to the corresponding fact for $\text{Hom}$: If $M, N$ are both left $R$-modules, say, then in general $\text{Hom}_R(M, N)$ has no natural $R$-module structure, but as we’ve seen it does have one when $R$ is commutative.

Remark. The tensor product is exactly analogous to the balanced product of $G$-sets. Moreover, the permutation representation functor $G\text{-set} \to \text{FG-mod}$ given by $X \mapsto FX$ is compatible with the two constructions: If $X$ is a right $G$-set and $Y$ a left $G$-set, then there is a natural isomorphism of $F$-modules $F(X \times_G Y) \cong FX \otimes_{FG} FY$.

Let’s note right away that this new tensor product can behave very differently from tensor product over $F$. In particular, a tensor product of two nonzero modules can be zero. For example, suppose $R$ is a principal ideal domain (e.g. $F[x]$) and $M = R/(a)$, $N = R/(b)$ with
a, b relatively prime. Then by construction $M \otimes_R N$ is annihilated by both $a$ and $b$, hence is zero.

We next consider various formal properties, beginning of course with a universal property. If $V$ is any $F$-module, an $F$-bilinear map $\beta : M \times N \to V$ is $R$-balanced if $\beta(xr, y) = \beta(x, ry)$ for all $r, x, y$. For example the natural map $\alpha : M \times N \to M \otimes_R N$ given by $\alpha(x, y) = x \otimes y$ is $R$-balanced, by the very definition. Note that an $R$-balanced map is $F$-bilinear.

**Proposition 1.1** If $\beta : M \times N \to V$ is $R$-balanced, then there is a unique $F$-module homomorphism $\overline{\beta} : M \otimes_R N \to V$ such that the following diagram commutes:

![Diagram](image)

*Proof:* By the universal property of tensor product over $F$, there is a unique $\gamma : M \otimes_F N \to V$ commuting in the appropriate diagram. It is then clear that $\gamma$ factors uniquely through $M \otimes_R N$, yielding the desired $\overline{\beta}$.

As in the case of $\otimes_F$, our tensor product is a functor in both variables. The precise way to say this is that $(M, N) \mapsto M \otimes_R N$ is a functor from the product category $\text{mod-}R \times \text{mod-}R$ to $\text{F-mod}$. Thus given module homomorphisms $\phi : M_1 \to M_2$ and $\psi : N_1 \to N_2$, we get an induced homomorphism

$$\phi \otimes_R \psi : M_1 \otimes_R N_1 \to M_2 \otimes_R N_2$$

where $(\phi \otimes_R \psi)(x \otimes y) = \phi(x) \otimes \psi(y)$.

**Proposition 1.2** $\otimes_R$ distributes over direct sums in both variables, up to natural isomorphism.

The meaning and the proof of the theorem are left to the reader; compare the corresponding fact for tensor products over $F$.

Next we’d like to say that tensor product is associative, but this doesn’t even make sense because of the left/right dichotomy. But with some extra structure we can do it, as we digress to explain.

### 1.1 Bimodules

Suppose $R, S$ are rings. An $(R, S)$-bimodule is an abelian group $M$ equipped with left $R$-module and right $S$-module structures (with the same underlying abelian group) such that $r(xs) = (rx)s$ for all $r \in R, s \in S, x \in M$. When $R, S$ are $F$-algebras, which is the case we are concerned with here, unless otherwise specified we require further that the $F$-module
structures obtained from $R$ and $S$ are the same. The main example we need at the moment is the following:

**Example:** Let $R$ be an $F$-algebra. Then $R$ is an $(R, R)$-bimodule under left and right multiplication in $R$; the bimodule condition is then equivalent to the associativity of multiplication. More generally if $A, B$ are subalgebras of $R$, then by restriction $R$ becomes a $(A, B)$-bimodule. More generally still, suppose $A, B$ are $F$-algebras and we are given homomorphisms $\phi : A \rightarrow R$, $\psi : B \rightarrow R$. Then $R$ is an $(A, B)$-bimodule with $a \cdot r \cdot b = \phi(a)r\psi(b)$.

The utility of this structure comes from:

**Proposition 1.3** Suppose $M$ is an $(R, S)$-bimodule and $N$ is a left $S$-module. Then $M \otimes_S N$ has a natural left $R$-module structure, given by $r(x \otimes y) = (rx) \otimes y$.

**Proof:** One has only to check that the given formula yields a well-defined map $R \times (M \otimes_S N) \rightarrow M \otimes_S N$ (use universal properties); then the conditions for an $R$-module follow trivially.

**Example and definition.** Suppose $S \subset R$ is a subalgebra. Then we get a functor $S$-mod $\rightarrow$ $R$-mod by treating $R$ as an $(R, S)$-bimodule and for any $S$-module $M$ forming the $R$-module $R \otimes_S M$. In particular we can take $R = FG$ for a group $G$ and $S = FH$ for a subgroup $H$. If $V$ is a representation of $H$, then $FG \otimes_{FH} V$ is the induced representation of $G$, often denoted $Ind_H^G V$. These are studied in detail in Serre §7, §8, and below.

Induced representations are exactly analogous to induced $G$-sets. In fact if we consider the permutation representation functor $X \mapsto FX$ (where $X$ is a $G$-set or $H$-set as appropriate), then for an $H$-set $X$ we have a natural isomorphism $F(G \times_H X) \cong FG \otimes_{FH} FX$ (check this!). In particular, if $X$ is the trivial one-element $H$-set, so $FH = F$ with trivial $H$-action, we see that $Ind_H^G F$ is just the permutation representation defined by $G/H$.

**Remark.** If $M$ is a left $R$-module, $R \otimes_R M$ is naturally isomorphic to $M$. The proof is identical to the proof for fields given earlier. Combining this with the fact that tensor products distribute over direct sums, we see that if $M$ is a free right $R$-module with basis indexed by a set $I$, then $M \otimes_R N \cong \oplus_I N$. (A similar result holds for $N$ a free left $R$-module.)

An important example: When $H$ is a subgroup of $G$, $FG$ is a free right $FH$-module, with basis given by any set of left coset representatives in $G/H$. Hence as $F$-modules an induced representation can be written

$$FG \otimes_{FH} V = \oplus_{g \in G/H} gV,$$

where the abusive notation $g \in G/H$ means that $g$ ranges over a fixed but arbitrary set of coset representatives.

We can also give the promised associativity result.

**Proposition 1.4** Suppose $L$ is a right $R$-module, $M$ is an $(R, S)$-bimodule, and $N$ is a left $S$-module. Then there is a natural isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N).$$
This can be proved similarly to the corresponding statement for $F$-modules.

Now suppose $\phi : S \to R$ is an $F$-algebra homomorphism. If $N$ is a left $R$-module, we write $\phi^*N$ for $N$ regarded as a left $S$-module via $\phi$. We call this “pullback along $\phi$” (not to be confused with other categorical uses of the term “pullback” that you may encounter). If $M$ is a left $S$-module, we denote by $j$ the $S$-module homomorphism $M \to R \otimes_S M$ given by $j(x) = 1 \otimes x$.

**Proposition 1.5** Suppose $M$ is a left $S$-module, $N$ is a left $R$-module, and $f : M \to N$ is a homomorphism of left $S$-modules. Then there is a unique homomorphism of $R$-modules $\overline{f} : R \otimes_S M \to N$ such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{j} & & \downarrow{\exists \overline{f}} \\
R \otimes_S M & & 
\end{array}
\]

**Proof:** As usual in these situations, the motto “uniqueness yields existence” applies. There is no choice but to define $\overline{f}(r \otimes x) = rf(x)$. Don’t forget that $r \otimes x$ is an equivalence class in a quotient module, and that a general element of a tensor product is a sum of elements of the form $r \otimes x$. So it would be a pain in the neck to check the well-definedness directly. Instead use the universal property of tensor products (as we did for tensor products over $F$) to define $\overline{f}$, then check after the fact that it satisfies the above formula.

I’ll work this out to remind you how it goes. We have an $S$-balanced map $R \times M \to N$ given by $\alpha(r, x) = rf(x)$. So $\alpha$ factors uniquely through an $F$-linear map $\beta : R \otimes_S M \to N$ (see the universal property above), and we take $\overline{f} = \beta$. Clearly $\overline{f}$ is an $R$-module map, the diagram commutes, and by construction $\overline{f}(r \otimes x) = rf(x)$.

**Plain English version:** If you want to define an $R$-module homomorphism $R \otimes_S M \to N$, it suffices (indeed is equivalent) to define an $S$-module homomorphism $M \to N$.

**Adjoint functor version:** The functor $R \otimes_S (-) : \textbf{S-mod} \to \textbf{R-mod}$ is left adjoint to the pullback functor $\phi^* : \textbf{R-mod} \to \textbf{S-mod}$. In other words, there is a natural bijection

\[
\text{Hom}_R(R \otimes_S M, N) \cong \text{Hom}_S(M, \phi^*N).
\]

Often $\phi$ is just inclusion of a subalgebra. In that case pullback is just restriction. The main case of interest for us is $FH \subset FG$, where $H$ is a subgroup of $G$. In that case we write $\text{Res}_H^G(-)$ for the restriction functor, and $\text{Ind}_H^G$ (“induction”) for $FG \otimes_{FH} (-)$. We will return to this in §3.1 below, after a slight detour.
2 Pullback along a group homomorphism, outer automorphisms and representations

In this section $G$ is an arbitrary group (not necessarily finite), $F$ is any field, and representations are over $F$. Moreover we need not even assume our representations are finite-dimensional; all the constructions below are quite formal.

Suppose $\phi : G' \rightarrow G$ is a homomorphism of groups, and $\rho_V : G \rightarrow GL(V)$ is a representation of $G$. Recall that by precomposition with $\phi$ we can define “pulled back” representation $\phi^*V$ of $G'$: The underlying vector space is just $V$ again, with $\rho_{\phi^*V} = \rho_V \circ \phi$. This yields a functor $\phi^*$ from $FG$-modules to $FG'$-modules. Once again, do not confuse this use of the term “pullback” with other categorical uses you may encounter.

Some simple properties of pullback:

- If $\phi$ is surjective, $\phi^*$ preserves irreducible representations.
- Pullback commutes with composition in the sense that if $G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3$ and $V$ is a representation of $G_3$, then $(\psi \circ \phi)^*V = \phi^*(\psi^*V)$.
- If $i : H \subset G$ is an inclusion, then $i^*$ is just restriction, also denoted $Res^G_H$.

The second item above implies that for any commutative diagram

\[
\begin{array}{ccc}
G' & \xrightarrow{\phi} & G \\
\downarrow{\xi'} & & \downarrow{\xi} \\
H' & \xrightarrow{\psi} & H
\end{array}
\]

representation $V$ of $H$, we have $(\xi')^*\psi^*V = \phi^*\xi^*V$.

Taking $G' = G$ and $\phi$ an automorphism, we see that the group of automorphisms $Aut G$ acts on the right of the set $Rep_F G$ of isomorphism classes of representations (here we reserve our notation $Rep_F G$ for finite-dimensional representations, although finite-dimensionality is irrelevant for the present constructions), preserving dimension and irreducibility. So we have a right action on the set of isomorphism classes of irreducible representations. If $\theta \in Aut G$, we sometimes call the pullback $\theta^*V$ the “twist” of $V$ by $\theta$. The following lemma is as important as it is trivial.

Lemma 2.1 If $\theta : G \rightarrow G$ is an inner automorphism, and $V$ is any representation of $G$, then $\theta^*V \cong V$.

Proof: Suppose $\theta(g) = xgx^{-1}$. Then $\rho_{\theta^*V} = \rho(x)\theta_V\rho(x)^{-1}$, so $\rho(x)$ provides the desired isomorphism.

Remark. In the case of finite-dimensional complex representations, this is also clear from character theory, since the trace is conjugation invariant.
Now let $\text{Inn} G \subset \text{Aut} G$ denote the inner automorphisms. It is clear that $\text{Inn} G$ is a subgroup, and trivial to check that it is a normal subgroup: Indeed for $x \in G$ let $c_x$ denote left conjugation by $x$; then if $\theta$ is any automorphism, $\theta c_x \theta^{-1} = c_{\theta(x)}$. We then define $\text{Out} G$, the “group of outer automorphisms” by $\text{Out} G = (\text{Aut} G)/(\text{Inn} G)$. Note the terminology is potentially misleading, since an element of $\text{Out} G$ is not an automorphism but rather a coset in $\text{Aut} G$. Somewhat inconsistently, we nevertheless refer to any automorphism that is not inner as an “outer automorphism”. With attention to context, no confusion should result.

**Examples.** Here we take $F = \mathbb{C}$ and finite-dimensional representations.

1. Take $G = A_4$ and let $\theta$ be the automorphism given by conjugating by a transposition in $S_4$. Note that although this would be an inner automorphism in $S_4$, it is an outer automorphism of $A_4$ because it exchanges the two conjugacy classes of elements of order 3. From a glance at the character table (Serre, p. 42) it is clear that $\theta$ also exchanges the two non-trivial 1-dimensional representations. Notice however that for the 3-dimensional irreducible $W$ of $A_4$, $\theta^* W \cong W$ by default, because $\theta^* W$ is irreducible and there is no other option. In fact a simpler example of the same phenomena is given by $A_3 = C_3 \subset S_3$.

2. If $G$ is abelian then $\text{Out} G = \text{Aut} G$. Taking $G = C_n^p$ for a prime $p$, we have $\text{Aut} G = GL_n F_p$, and the action on the $p^n - 1$ non-trivial 1-dimensional representations is transitive.

### 3 Induced representations

This section is a commentary on §7 and §8 of Serre. It will be expanded on further in the lectures. We often write $\text{Ind}^G_H M$ for $FG \otimes_{FH} M$ ($H$ a subgroup of $G$).

#### 3.1 The adjoint functor property of induced representations

This property is treated in an off-hand way in Remark (2) of §7.1. I would take it as the fundamental property of induced representations. Frobenius reciprocity, expressed directly in terms of modules, is just another manifestation of the adjoint property. The character version is a corollary.

I first restate the adjoint functor property (as discussed above under tensor products) for this particular case. But we may as well do it for arbitrary $F, G$ and arbitrary $FG$-modules.

**Proposition 3.1** Let $M$ be a left $FH$-module, $N$ a left $FG$-module.

There is a natural isomorphism of $F$-modules

$$\text{Hom}_{FG}(FG \otimes_{FH} M, N) \xrightarrow{\cong} \text{Hom}_{FH}(M, N).$$

In other words, $\text{Ind}^G_H$ is left adjoint to $\text{Res}^F_H$.

From the general $F$-algebra version we see explicitly how this isomorphism is defined: Given $\phi : FG \otimes_{FH} M \rightarrow N$, we precompose with $j : M \rightarrow FG \otimes_{FH} N$ (given by $j(x) = 1 \otimes x$) to get a homomorphism of $FH$-modules $\psi : M \rightarrow N$. Given a homomorphism of $FH$-modules $\psi : M \rightarrow N$, we get a homomorphism of $FG$-modules $\phi : FG \otimes_{FH} M \rightarrow N$ by $\phi(g \otimes x) = g\psi(x)$. 

6
3.2 Mackey’s double-coset formula

By this I mean Proposition 22. Managing all the subgroups and conjugations in this formula can get confusing, so it may be helpful to think about in a slightly different way, emphasizing pullbacks of representations as discussed above. In my view Serre’s choice of notation is less than optimal, so I’m going to modify it and even reverse it; please read carefully.

My definition:

$$K_x = xHx^{-1} \cap K \quad H_x = x^{-1}Kx = H \cap x^{-1}Kx.$$ 

Thus Serre’s $H_s$ would be my $K_s$. I submit that the above notation is better for two reasons: First of all, in my notation $K_x$ is a subgroup of $K$ and $H_x$ is a subgroup of $H$. Second, the new notation is consistent with our notation for isotropy groups: $K_x$ is the isotropy group of $xH$ for the left action of $K$ on $G/H$, while $H_x$ is the isotropy group of $Kx$ for the right action of $H$ on $K\setminus G$.

Now for any $g \in G$ let $c_g$ denote left conjugation by $g$. Then in my notation the Mackey double-coset formula reads:

$$Res^G_H Ind^K_{K_x} W \cong \bigoplus_{x \in K \setminus G/H} Ind^K_{K_x} c_{x^{-1}}^* Res^K_H W.$$ 

Here we will also use Serre’s abbreviation: $W_x = c_{x^{-1}}^* Res^K_H W$. The convention for interpreting $x \in K \setminus G/H$ is that $x$ ranges over a set of $(K,H)$ double-cosets (again I reverse Serre’s notation—why not keep $K,H$ in the same order?), the particular choice being immaterial. For example, the fact that $Ind^K_{K_x} W_x$ depends (up to isomorphism) only on the double-coset of $x$ ultimately comes from Lemma 2.1. In fact showing that it is invariant under replacing $x$ by $xh$ is an easy consequence of that Lemma and our other simple properties of pullback. Showing that it is invariant under replacing $x$ by $kx$ needs a further lemma describing the behaviour of induction under pullback along an isomorphism, as we next discuss.

Suppose we have a commutative diagram of groups

$$
\begin{array}{ccc}
H' & \xrightarrow{\psi} & H \\
\downarrow{i'} & & \downarrow{i} \\
G' & \xrightarrow{\phi} & G
\end{array}
$$

in which $\phi$ is an isomorphism and $i$ is inclusion of a subgroup, $H' = \phi^{-1}H$ and $i'$ is inclusion. Thus $\psi$ is just the restriction of $\phi$ to $H'$.

**Proposition 3.2** Let $M$ be an $FH$-module. There is a natural isomorphism
\[ \phi^* \text{Ind}_H^G M \cong \text{Ind}_H^{G'} \psi^* M. \]

Proof sketch. The main step is to find a plausible candidate homomorphism

\[ \text{Ind}_H^{G'} \psi^* M \rightarrow \phi^* \text{Ind}_H^G M. \]

For this it is enough to define a homomorphism of \( FH' \)-modules \( \psi^* M \rightarrow \phi^* \text{Ind}_H^G M \); just take the standard inclusion \( M \rightarrow \text{Ind}_H^G M \) (thought of as a homomorphism of \( FH' \)-modules). Then check it works.

Corollary 3.3 Suppose \( G' = G \) and \( \phi \) is an inner automorphism. Then \( \text{Ind}_H^{G'} \psi^* M \cong \text{Ind}_H^G M \).

The invariance under “replacing \( x \) by \( kx \)” follows from the corollary.

Keep in mind also the useful and much easier special case of the Mackey formula where \( H \) is normal and \( K = H \). Then the double-cosets are just the ordinary cosets \( G/H \), \( H_s = H \) for all \( s \), and \( \text{Ind}_{H_s}^H W_s = W_s = c_{s^{-1}}^* W \). Note that \( c_{s^{-1}} \) is by definition an inner automorphism of \( G \), but need not be an inner automorphism of \( H \). Hence \( W_s \) is not necessarily isomorphic to \( W \).

### 3.3 Mackey’s irreducibility criterion

I’ll give a slightly different statement and proof of the Mackey criterion. Since tensor products distribute over direct sums, it is clear that if \( W \) is reducible then so is \( \text{Ind}_H^G W \). So we may as well assume \( W \) is irreducible from the beginning. Then the Mackey criterion can be stated as follows (\( H_s \) and \( W_s \) are as in Serre, Prop. 22):

**Theorem 3.4** Let \( W \) be an irreducible representation of \( H \), and let \( s \) range over a set of double-coset representatives for \( H \backslash G/H \), taking \( s = 1 \) for the double-coset \( H \). (the choice won’t matter). Then \( \text{Ind}_H^G W \) is irreducible if and only if for all \( s \neq 1 \), \( W_s \) and \( \text{Res}_{H_s}^H W \) are disjoint as \( H_s \)-representations.

Proof: \( \text{Ind}_H^G W \) is irreducible if and only if \( \dim \text{End}_{GH}(\text{Ind}_H^G W) = 1 \). By the adjoint functor property, this is equivalent to

\[ \dim \text{Hom}_{GH}(W, \text{Ind}_H^G W) = 1. \]

In the double-coset formula (Prop. 22, with \( K = H \)) the summand with \( s = 1 \) has \( H_s = H \) and \( W_s = W \), so contributes 1 to the dimension of the displayed \( \text{Hom} \) above (since \( W \) is irreducible as \( GH \)-module). So the displayed equation is equivalent to

\[ \dim \text{Hom}_{CH}(W, \text{Ind}_H^H W_s) = 0 \]

for all \( s \neq 1 \). On the other hand, since \( W \) is irreducible and all our modules are completely reducible, we can reverse the source and target of the above \( \text{Hom} \) without changing its...
dimension. So by another application of the adjoint functor property we get the equivalent condition

$$\dim \text{Hom}_{CH_s}(W_s, \text{Res}^H_{H_s} W) = 0.$$ 

Since all representations are completely reducible here, this is equivalent to the disjointness assertion.

The main point here is that in the criterion as stated by Serre, we actually only need to check $s$ ranging over a set of $H - H$ double-coset representatives. This makes some applications much easier, for example the exercise on $SL_2$ below. On the other hand, sometimes it’s just as easy to check a general $s$, which has the advantage that you don’t need to strain your brain thinking about double-cosets. This is the case, for example, in the Wigner-Mackey construction of the following section.

Two important special cases to keep in mind:
1. $H$ is a normal subgroup. In this case both the statement (see text) and the proof (do it directly yourself) are easier. The point is that the double-cosets are just the ordinary cosets, $H_s = H$ for all $s$, and one has only to show that $W$ and $c^*_s W$ are distinct.
2. $W$ is 1-dimensional. Then $W_s$ and $\text{Res}^H_{H_s} W$ are given by homomorphisms $H_s \rightarrow \mathbb{C}^\times$, and one only needs to check whether they are identical or not.

### 3.4 Semi-direct products by an abelian group

This is §8.2 in Serre, the “Wigner-Mackey method”. Although much verbiage is required to state and prove the result, it’s quite elegant and gives a constructive classification of all irreducibles for this particular class of groups. It helps to work out a specific example where we already know the answer by other means. I’ll use the notation $K^\#$ for $\text{Hom}(K, \mathbb{C}^\times)$ ($K$ any finite group).

Let’s take $G = S_4$, thought of in our now standard way as the semidirect product $A \rtimes S_3$ with $A \cong C_2^2$, and construct the irreducibles using only the Wigner-Mackey approach. I’ll use Serre’s notation for the groups and representations (or characters) involved. Recall also that our default embedding of $S_{n-1}$ in $S_n$ is always as the permutations fixing $n$.

1. We start with the action of $H = S_3$ on $A^\#$ and determine the orbits. There are two: the trivial homomorphism is a fixed point, while $S_3$ transitively permutes the three non-trivial homomorphisms.
2. We choose representatives from each orbit. In the first case there is only one choice, the trivial homomorphism $\chi_1$. There’s no great advantage in making an explicit choice of $\chi_2$ from the second orbit, but if one is wanted I’ll take the following: $A$ is generated by the free involutions (12)(34) and (13)(24). Let $\chi_2 \in A^\#$ be the non-trivial homomorphism with kernel generated by (12)(34).
3. We determine the isotropy group $H_i \subset H$ of $\chi_i$, let $G_i = A \rtimes H_i$, and extend $\chi_i$ to $G_i$ by setting $\chi_i(h) = 1$ for $h \in H_i$ (you should check that this does yield a homomorphism, thanks to the fact that $H_i$ fixes $\chi_i$). For $\chi_1$ we always get $H_i = H$ and $G_i = G$, so $\chi_1$ is just the trivial homomorphism $S_4 \rightarrow \mathbb{C}^\times$. For $\chi_2$ we get $H_2$ of order 2; with the explicit choice
above, $H_2$ is generated by (12). Then $G_2$ is a 2-Sylow subgroup of $S_4$ (isomorphic to the dihedral group).

4. We take an irreducible representation $\rho$ of $H_i$, pull it back to $G_i$, tensor with $\chi_i$ and induce up to $G$. Then according to Prop. 24, these representations are irreducible, distinct, and form a complete list of the irreducibles. In the case of the trivial character $\chi_1$, we are just pulling back the irreducibles of $S_3$ and tensoring with the trivial representation, so the construction is just giving us the irreducibles pulled back from $S_3$ (this happens in general), of which there are three. In the case of $\chi_2$, we can pull back either the trivial or sign representation of $H_2 = C_2$ to $G_2$, tensor it with $\chi_2$ and then induce it up to $S_4$. So we get two irreducible reps of dimension 3, and have constructed the five irreducibles of $S_4$. One could go further and check how these two 3-dimensional reps match up with the constructions we already have, but I’ll leave that to the interested reader.

4 Serre §8.3-5

We’ve already done most of this material (nilpotent and solvable groups, Sylow theorems). Even “supersolvable groups” made a brief appearance in earlier notes. The main new item is the elegant Theorem 16. This is as far as we’ll go in Serre, at least for now.

5 Exercises

1. We’ve been neglecting the quaternion group for too long. In this exercise we’ll make up for it. I’ll use $Q$ to denote the quaternion group of order 8. All representations are over C.

   a) Determine the irreducible representations of $Q$. You will find one 2-dimensional representation, an explicit description of which appears in any number of sources. But don’t go peeking at such sources. Instead, describe it as an induced representation. Also give the character table.

   b) Determine $\text{Aut} Q$, $\text{Inn} Q$ and $\text{Out} Q$. By “determine” I mean show that they are isomorphic to certain familiar groups.

   Note: The small order of $Q$ notwithstanding, it can get a bit messy to check rigorously that a given map $Q \rightarrow Q$ really is a homomorphism. Those who have taken Manifolds have presumably seen “presentations of groups”, which can be used to greatly simplify matters. Alas, we have not had time for this, so you should prove for yourself and use the following ad hoc substitute here (you don’t need to turn in the proof of the lemma).

   Lemma. Let $G$ be any group, and let $a, b \in G$ be elements satisfying (i) $a^4 = 1$, (ii) $a^2 = b^2$ and (iii) $aba^{-1} = b^{-1}$. Then there is a unique homomorphism $\phi : Q \rightarrow G$ such that $\phi(i) = a$, $\phi(j) = b$.

   (If you do know about presentations, show instead that $Q$ has presentation $\langle i, j : i^4, i^2j^2, ijij^{-1}j \rangle$. The lemma is then immediate from this.)

   c) Describe explicitly the action of $\text{Out} Q$ on the irreducibles found in (a).
d) Generalized quaternion groups. Regard $\mathbb{C} \subset \mathbb{H}$ as usual, we have $\mathbb{C}^\times \subset \mathbb{H}^\times$. Then $Q$ is the subgroup generated by (i) the 4th roots of unity in $\mathbb{C}^\times$, and (ii) the element $j$. We can generalize this construction by replacing the 4th roots of unity by $m$-th roots of unity for any $m$, with the term generalized quaternion group being reserved for the case $m = 2^n$. Let’s use the notation $Q_n$ for the group generated by $2^n$-th roots of unity and $j$. We can generalize this construction by replacing the 4th roots of unity by $m$-th roots of unity for any $m$, with the term generalized quaternion group being reserved for the case $m = 2^n$. Let’s use the notation $Q_n$ for the group generated by $2^n$-th roots of unity and $j$. It has order $2^n+1$ and fits into a non-split extension $C_{2^n} \rightarrow Q_n \rightarrow C_2$. Note this is a departure from earlier notation; here $Q = Q_2$ whereas formerly I wrote $Q_8$ for $Q$.

d1. Determine the irreducible representations of $Q_n$ for all $n$, in the following form: Since $Q_n$ is nilpotent and hence supersolvable, we know that every irreducible is induced from a 1-dimensional rep of a subgroup. Describe the irreducibles explicitly as such induced reps, i.e. give the subgroup in question and the 1-dimensional rep explicitly.

d2. Here’s one way generalized quaternion groups come up: Show that if $p$ is an odd prime, then the 2-Sylow subgroup of $SL_2 \mathbb{F}_p$ is generalized quaternion. (You’ll need to separate into the cases $p = 1 \mod 4$ and $p = 3 \mod 4$, with the first case being much easier.) If you’re comfortable with finite fields $\mathbb{F}_q$, do it for that case where the same result holds ($q$ odd).

A cultural aside. The generalized quaternion groups also arise in topology, in the “spherical space-form” problem. This very difficult problem (solved in 1978) asks the question: Which finite groups $G$ can act freely on a sphere $S^n$? Here we mean action by homeomorphisms. Then the orbit space $S^n/G$ is a “spherical space-form”, i.e a manifold with $S^n$ as its universal cover. (In the original version of the problem, one asks that $G$ act via isometries in the standard Riemannian metric on $S^n$. Then the covering map $S^n \rightarrow S^n/G$ is a local isometry, which helps to explain the terminology.) For example, any cyclic $G$ acts freely on $S^n$ for $n = 2m - 1$ odd, by thinking of $S^n$ as the unit sphere in $\mathbb{C}^m$ and using scalar multiplication by roots of unity.

To simplify matters, I’ll only discuss the case when $G$ is a finite $p$-group. Then for $p$ odd, the cyclic $p$-groups are the only $p$-groups that can act freely on a sphere. If $p = 2$, however, we can replace $\mathbb{C}$ by $\mathbb{H}$ in the preceeding paragraph and take $n = 4m - 1$. Then we can think of $S^n$ as the unit sphere in $\mathbb{H}^m$, and use scalar multiplication as before to get a free action of any generalized quaternion group on $S^n$. The punchline is that a finite 2-group acts freely on some sphere if and only if it is cyclic or generalized quaternion.

2. Exercise 7.4 in Serre. (Remember that you only need to check the conditions of the Mackey criterion on a set of double-coset representatives (excluding the identity double-coset). There is only one double coset to check! (See the discussion of $SL_n$ and $SL_2$ in the $G$-sets notes.)

3. The Mackey double-coset formula describes induction followed by restriction. What happens for restriction followed by induction? Answer this in the special case of $\text{Ind}^G_H \circ \text{Res}^G_H$ by showing that if $M$ is an $FG$-module, then there is a natural isomorphism of $FG$-modules

$$FG \otimes_{FH} M \xrightarrow{\sim} F(G/H) \otimes_F M,$$
where on the right I mean the tensor product of representations defined last quarter, with
\( g \cdot (aH \otimes x) = (gaH \otimes gx) \). Last quarter we were vague about the word “natural”, but this
time you should prove that you have a natural transformation of functors (see the Category
Theory notes).
Finally, conclude that for any \( FG \)-module \( M \), \( FG \otimes_F M \) (always with the diagonal action
unless otherwise mentioned) is a free \( FG \)-module.

4. The restriction functor \( Res^G_H \) obviously commutes with tensor products of representa-
tions, i.e. if \( W, V \) are representations of \( G \), then \( W \otimes_F V \) restricted to \( H \) is the same thing
as restricting \( V, W \) to \( H \) and then tensoring. But the analogous statement for induction is
clearly false, since the dimensions don’t even match. What we get instead is this: Suppose
\( W \) is an \( FG \)-module and \( V \) is an \( FH \)-module. Then there is a natural isomorphism of
\( FG \)-modules

\[
\text{Ind}^G_H(V \otimes_F Res^G_H W) \cong (\text{Ind}^G_H V) \otimes_F W.
\]
Prove this (you can omit the proof of naturality). It is discussed and perhaps even
proved (in a low-tech way) at various points in Serre, but do not use the text arguments.
Use universal properties of tensor products, induction etc. to define your maps.
The preceeding problem is a special case of this result. Why/how?

5. If \( I \subset R \) is a 2-sided ideal and \( M \) is an \( R \)-module (say a left module), then we have
see that \( M/IM \) is an \( R \)-module and \( M \rightarrow M/IM \) is an \( R \)-module homomorphism.
a) Suppose \( R \) is an \( F \)-algebra. Show that there is a natural isomorphism \( (R/I) \otimes_R M \cong
M/IM \). Prove the naturality; in other words, show that your isomorphism is a natural
transformation between the appropriate functors \( \textbf{R-mod} \rightarrow (R/I)-\text{mod} \).
b) Take \( R = FG \) and \( I = IG \) the augmentation ideal (i.e. the kernel of the augmentation
\( \varepsilon : FG \rightarrow F \)).
b1) Show that \( F \otimes_{FG} M \) is the maximal trivial quotient of \( M \) (part of your job being to
make precise “maximal trivial quotient”).
b2) Conclude that if \( M \) is finite-dimensional and completely reducible, then \( \text{dim}_F(F \otimes_{FG}
M) \) is the multiplicity of the 1-dimensional trivial representation in \( M \).

6. Let \( G = U_{3p} \), the group of upper triangular unipotent matrices in \( GL_{3p} \). Use the
Wigner-Mackey method to simultaneously construct and classify the irreducible complex
representations of \( G \). For this purpose you should consider \( G \) as the semidirect product
\( H \rtimes K \), where \( H \) is the “right column subgroup” and \( K = U_{3p} \).
Note: Use only the Wigner-Mackey method. You don’t need to look at conjugacy classes,
the sum of squares formula, divisibility theorems or any of that. The beauty of Wigner-
Mackey is that it uses a single construction to give a complete answer. (Although you will
probably want to check your answer after the fact to be sure it agrees with our earlier results!)