Homework 5 due Feb. 13

Reading: “Galois extensions” and “The fundamental theorem of Galois theory”.

1. Suppose $f$ is an irreducible quartic over $\mathbb{Q}$ with exactly two real roots. Show that its Galois group (i.e. the Galois group of its splitting field over $\mathbb{Q}$) is either $D_8$ or $S_4$.

   Note: This is a fun problem that doesn’t use anything beyond the pre-midterm material. It occurs much later in the text, for some reason.

2. Formulate and prove a “recognition principle” for the Galois group to be a semi-direct product. Your theorem should fit into the following template: Let $E/F$ be a finite Galois extension with Galois group $G$, and let $K/F$ be a subextension that is also Galois over $F$. (So $G(E/K)$ is normal and we have an extension $G(E/K) \rightarrow G(E/F) \rightarrow G(K/F)$ by an earlier theorem.) Then the extension splits if and only if there is a subextension $L/F$ such that...you fill in the blank!

3. Do exercise 89 on p. 82, either with the hint or instead using Artin’s theorem directly (symmetric functions are by definition the fixed field of the evident $S_n$-action; be sure to justify that this action really is via field automorphisms). Then do exercise 91, whose “hint” is pretty much the complete one-liner proof. Although this all makes for an almost trivial exercise, it is an elegant result, showing that every finite group can be realized as a Galois group of some Galois extension.

4. Exercises 92, 93, 94.