Why representation theory?

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Why is representation theory important? The question was raised in class, and it's a good one. Why indeed? It's not so clear when you first encounter the subject. In this essay I hope to shed some light on the matter. In order to do so I must ask you to grant me some poetic license, as it won't always be possible to give precise statements or even to define my terms precisely.

In a nutshell, there are two main reasons why representation theory is so important:

I. Representations can help us understand a particular group, or a whole class of groups.

II. Representations arise in a wide variety of contexts. We may be faced with a particular representation V that we need to understand.

I proceed to elaborate.

I. The first reason is simply that often one can better understand a particular group, or a whole class of groups, by looking at representations. After all, we understand matrices very well, so if we can "represent" the group G inside the invertible matrices it may help us to understand G. But before illustrating this, let's look at a simpler analogue that you've already studied extensively.

Representations in a permutation group. Take G to be finite. As we've seen, the study of finite G-sets is equivalent to the study of "permutation representations", i.e. homomorphisms $\phi : G \longrightarrow S_n$. We "represent" the group in S_n, regarding this as progress because permutations are concrete and comparatively easy to understand. By Cayley's theorem we can always get faithful representations (i.e. ϕ injective), thereby identifying G with some permutation group. For example D_8, the dihedral group of order 8, starts off in life as the group of symmetries of the square. But then one observes that it is completely determined by its action on the vertices and hence is a subgroup of S_4. Admittedly, this is not a very convincing example, but still the technique can be useful.

In fact, as we've seen, our permutation representation is often most useful when it is *not* faithful, because its kernel is a normal subgroup N; having an extension $N \longrightarrow G \longrightarrow G/N$ is definitely progress toward understanding G. We've seen many problems that can be attacked by looking for suitable G-sets, i.e. permutation representations and exploiting either their kernels as just mentioned or the isotropy groups of the action: Sylow theory, for example, or proving that groups of order pq are solvable.

Now, back to linear representations. A faithful representation $G \longrightarrow GL_n F$ gives us another way to understand G. For example, our dihedral group D_8 is isomorphic to the subgroup of $GL_2\mathbb{R}$ consisting of the matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

This is a very nice way to look at D_8. But of course, the real payoff comes from considering more complicated groups. For example, there is an infinite group called the "free product of C_2 and C_2", denoted $C_2 * C_2$ that comes up in topology as the fundamental group of a one-point union of two copies of the projective plane (if you know about such things). Its direct definition isn't so bad, but it has a beautifully simple faithful representation as the group of matrices

$$\begin{pmatrix} 1 & n \\ 0 & \pm 1 \end{pmatrix}$$

with $n \in \mathbb{Z}$.

As in the case of permutation representations, however, often we are just as interested in non-faithful representations. One beautiful application we'll see this quarter is to Burnside's $p^a q^b$-theorem. Here again we're trying to show certain finite groups G are solvable. Your first attempt would no doubt be to use Sylow theory or whatever you can cook up to produce a permutation representation whose kernel is a proper non-trivial subgroup. Then you have an extension whose kernel and quotient are groups of the same type with lower order, and you're done by induction (since solvable groups are closed under extensions). But this doesn't seem to work. However, a non-faithful (and non-trivial, of course) linear representation would serve just as well, and it is this method that Burnside ingeniously applies.

For another example, we turn to number theory. In some sense, large parts of number theory are equivalent to understanding the absolute Galois group of $\mathbb{Q}$, which I'll denote $G(\overline{\mathbb{Q}}/\mathbb{Q})$. Here $\overline{\mathbb{Q}}$ is the "algebraic closure" of $\mathbb{Q}$. I don't assume you've seen this concept, but it's easy to explain here: It is the subfield of $\mathbb{C}$ consisting of all algebraic numbers, that is, complex numbers that are roots of some polynomial with rational coefficients. It is a very complicated and mysterious group. For example, it isn't known which finite groups can occur as a quotient of it, or equivalently which finite groups can occur as the Galois group of some finite extension field of $\mathbb{Q}$. (Here I'm ignoring a technical point, namely that $G(\overline{\mathbb{Q}}/\mathbb{Q})$ has a certain topology and only those quotients obtained by modding out a *closed* normal subgroup are relevant.) One way to at least get some handle on this group is to find interesting representations of it. It's not at all obvious where you would even look for such representations, but number theorist have their ways and the method has been used with great success. Far from complete success, mind you, but this is what keeps mathematicians employed.

II. Another big reason that representation theory is important is that representations of groups very frequently pop up "in nature", in completely different contexts.

Here's an example from Galois theory: Suppose L/K is a finite Galois extension, with Galois group G. Then G acts on L by field automorphisms fixing K pointwise. This means that in particular it acts K-linearly on the K-vector space L, i.e. we have a representation of G over K, of dimension $|G|$. So what representation is it? Surely the answer would be enlightening. The first step is, as usual, to guess the answer—preferably using the celebrated *Optimist Principle*. So we look around to see if there are any obvious representations of G of dimension $|G|$ lying about, and indeed there is one: the regular representation, given by the group algebra KG itself as a left KG-module. This answer is so natural that it has to be right, and after checking an easy case or two such as $\mathbb{C}/\mathbb{R}$, you are ready to bet a large sum of money on it. You win your bet, but it takes some non-trivial work that we will carry out in Winter.

Representations come up all over the place in topology. Here I can only explain this in vague terms, unless you already know something about homology. Homology groups are certain functors from the category of topological spaces to the category of abelian groups. Note this automatically yields the assertion that homeomorphic spaces have isomorphic homology groups, the contrapositive of which already hints at the myriad applications. Moreover it is easy to modify the construction so that homology becomes a functor to F-vector spaces; this is called "homology with coefficients in F". Here F could be any field, but in practice it is usually $\mathbb{F}_p$ for some prime or $\mathbb{Q}$. Now, very often we are interested in continuous group actions on topological spaces; picture, for instance, C_2 acting on a sphere by $x \mapsto -x$, or the circle group S^1 acting on the two-sphere by rotation around the z-axis. On the other hand, it follows trivially from the axioms that functors take G-actions to G-actions: A G-action on an object X in some category is a homomorphism $G \longrightarrow Aut\, X$, and a functor F induces a group homomorphism $Aut\, X \longrightarrow Aut\, F(X)$; now compose. So if X is a space with G-action then its homology with coefficients in F yields representations of G! This can yield tons of useful information. For example there was a major industry in the 1980's (including my thesis and most of my early research) that involved translating decompositions of these representations into "decompositions" (sorry, must be vague here) of the spaces themselves. This was of interest for the usual reason, among others, namely understanding a complicated object by breaking it into smaller pieces.

In differential topology (one of my favorite subjects, lying between algebraic topology and differential geometry) the group actions of interest are "Lie group" actions on "smooth manifolds", which anyone taking the manifolds course will learn about in great detail. Meanwhile picture the rotation action of the circle group on the sphere to get the idea. In this setting if you have a fixed point (such as the poles in the example) the differential (=derivative) of your action will induce a representation of G on the tangent space at that point. This is yet another instance of the time-honored strategy of linearization. If the damn thing is too complicated, linearize it! In the case of compact Lie group actions this works exceptionally well, as one can show that in a neighborhood of the fixed point the group action is "locally isomorphic" to the representation on the tangent space (again, think of our example, where this is pictorially extremely plausible).

As these last couple of examples illustrate, often one is faced with a particular representation V of G and would like to "identify" it. If we have in our back-pocket a complete classification of all representations of G over the given field F, it would certainly make this task easier; one can sometimes even proceed by process of elimination. This is one reason that it is of interest to prove such classification theorems. In the case of finite G and $F = \mathbb{C}$, we will in some sense "classify" all the representations, and at least make a start at carrying out the same program for $F = \mathbb{R}$ ($\mathbb{C}$ is always easier because it is algebraically closed!).

This list of examples could be continued *ad infinitum*, going through algebraic geometry, harmonic analysis, combinatorics, physics and many other fields. In fact if you look at the preface of Serre's text, you'll see that Part I was written for "quantum chemists". Representation theory is a great thing!