Representations of finite groups, I

November 30, 2014

This series of notes is a guide to reading Serre's text, with some additional results and exercises included. A key point to realize is that Part I of the text is aimed at readers who have much less background in abstract algebra than we do. So you need to constantly be alert to the possibility of simplifying Serre's proofs by applying more advanced machinery. In particular, Serre doesn't introduce the group algebra until Chapter 6, whereas we already have it and will exploit it every chance we get. The same remark applies to modules.

Notation: F is always a field. G is a finite group except where otherwise specified. We will use the notation $Rep_F^k G$ for the set of isomorphism classes of irreducible representations of G over F of dimension k.

There are certain notations in Serre that I won't use. For example, he uses g for $|G|$, and z^* for complex conjugates. We'll stick with $|G|$ and $\overline{z}$. On the other hand I will use his notation $(v|w)$ for the Hermitian inner product, or "scalar product" of two vectors. Normally I use $\langle v, w\rangle$ but Serre uses it for something else and this would lead to major confusion.

1 Serre, Chapter I

1.1 Completely reducible modules, semisimple rings, and Maschke's theorem

In our language, Serre's Theorem 1 says that every short exact sequence of finite dimensional $\mathbb{C}G$-modules splits. Now for any ring R, call an R-module M *completely reducible* if it is a finite direct sum of simple modules. Call R *semisimple* if every finitely-generated R-module is completely reducible. Then Serre's Theorem 2 says that every finite dimensional $\mathbb{C}G$-module is completely reducible, and hence $\mathbb{C}G$ is semisimple. In this section we consider more generally for which F, G the algebra FG is semisimple.

If V, W are FG-modules, then $Hom_F(V, W)$ is an FG-module, with $g \in G$ acting by $(g \cdot \phi)(v) = g\phi g^{-1}(v)$. (We note in passing that there is no analogue of this construction for general F-algebras; it is special to group algebras.) Moreover

$$(Hom_F(V, W))^G = Hom_{FG}(V, W).$$

The implication (c) $\Rightarrow$ (a) in the next theorem is Maschke's theorem.

Theorem 1.1 *Let F be a field, G a finite group. Then the following are equivalent:*
a) Every short exact sequence of finite dimensional FG-modules splits.
b) FG is semisimple.
c) $char\, F$ doesn't divide $|G|$.

Proof: (a) $\Rightarrow$ (b): Let M be a nonzero $\mathbb{C}G$-module. If M is irreducible, it is completely reducible. If not, then there is a short exact sequence $0 \longrightarrow V \longrightarrow M \longrightarrow M/V \longrightarrow 0$ with V irreducible and $M/V \neq 0$. By induction (on either dimension or length, take your pick) we can assume M/V is completely reducible. Since the short exact sequence splits, M is completely reducible as required.

(b) $\Rightarrow$ (c): Suppose p divides $|G|$. Then I claim the regular representation is not completely reducible. Consider the short exact sequence

$$0 \longrightarrow Ker\, \epsilon \longrightarrow FG \xrightarrow{\epsilon} F \longrightarrow 0.$$

Here F has the trivial G-action and $\epsilon(g) = 1$ for all $g \in G$. If FG is completely reducible, then by Schur's lemma some irreducible summand L of FG maps isomorphically to F under ϵ. But L must then be a trivial 1-dimensional module, and the only such submodule of FG is generated by $\overline{G}$. But since $p|\,|G|$, $\epsilon(\overline{G}) = 0$, contradiction.

(c) $\Rightarrow$ (a) Suppose $char\, F$ doesn't divide $|G|$. Then the averaging operator e_0 (our notation; see the notes on algebras)

$$e_0 = \frac{1}{|G|} \sum_{g \in G} g \in FG$$

is defined, and projects any module M onto its fixed-points M^G. In particular it projects $Hom_F(V, W)$ onto $Hom_{FG}(V, W)$. Now consider a short exact sequence (the modules need not be finite dimensional here)

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0.$$

Since every F-module is free, there is a splitting $s : N \longrightarrow M$ as vector spaces. Then $e_0 s : N \longrightarrow M$ is a map of FG-modules, and it is still a splitting:

$$\pi(e_0 s) = e_0(\pi s) = e_0 Id_N = Id_N.$$

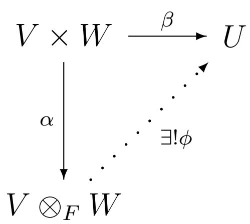
Thus Serre's Theorems 1 and 2 are special cases of the preceding theorem.

1.2 Tensor products

Serre gives a very "low-tech" version of tensor products, since he is writing for quantum chemists. It might be good enough for them, but it's not good enough for us; a more thorough treatment is needed. In fact tensor products can be defined for modules over an arbitrary ring R, but we will limit ourselves here to tensor products over fields F. Later we'll consider the case of modules over F-algebras, and later still modules over arbitrary rings.

Let V, W be vector spaces over F, not necessarily finite-dimensional. The tensor product $V \otimes_F W$ is defined by the following proposition:

Proposition 1.2 *There is an F-vector space $V \otimes_F W$ and a bilinear map $\alpha : V \times W \longrightarrow V \otimes W$ such that for any F-vector space U and bilinear map $\beta : V \times W \longrightarrow U$, there is a unique linear map $\phi : V \otimes_F W \longrightarrow U$ such that $\phi\alpha = \beta$:*



Moreover, $V \otimes_F W$ is unique up to a canonical isomorphism.

Proof: Let $F(V \times W)$ denote the F-vector space with basis $V \times W$. Then let $V \otimes_F W$ denote the quotient $F(V \times W)/X$, where X is the subspace spanned by all elements of the form $(cv, w) - c(v, w)$, $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$ and the analogues with the roles of v, w reversed. We write $v \otimes w$ for the equivalence class of (v, w) in $V \otimes_F W$. Setting $\alpha(v, w) = v \otimes w$, we see that α is bilinear by construction. Given β as above, we take $\phi(v \otimes w) = \beta(v, w)$. The bilinearity of β implies that ϕ is well-defined, and is the unique linear map commuting in the diagram.

The uniqueness statement follows by the universal argument used for universal properties. I'll remind you of the idea. If you have two object X, Y satisfying the same universal property such as the above, then the property itself formally yields unique morphisms $X \longrightarrow Y$ and $Y \longrightarrow X$ commuting in the appropriate diagram (the triangle above, in our present example). The composite of these two in either order yields the appropriate identity map, because it is a morphism commuting in the same diagram (the triangle in our case) as the identity map. In this sense the isomorphism between X and Y is not merely "canonical" (an undefined term), but actually unique, subject to the commutative diagram condition.

Remarks. 1. *Caution.* Note that not all elements of $V \otimes_F W$ are of the form $v \otimes w$. In general they are linear combinations of such elements: $\sum c_i(v_i \otimes w_i)$.

2. The Plain English version of the universal property says: If you want to define a linear map $V \otimes_F W \longrightarrow U$, it is enough (indeed equivalent) to define a bilinear map $V \times W \longrightarrow U$. There is an adjoint functor version too, but we'll save that for another day.

3. From a philosophical point of view, one advantage of the tensor product is that it allows us to treat bilinear maps in terms of morphisms in our original category **F-mod**. Instead of inventing a new category to deal with our bilinear β, we reinterpret it as a good old-fashioned linear map ϕ.

We develop some formal properties of the tensor product before going further (always a good idea). First of all, the tensor product is a functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ where $\mathcal{C}$ =**F-mod**. If

$f_i : V_i \longrightarrow W_i$ are linear maps for $i = 1, 2$, then $f_1 \otimes f_2 : V_1 \otimes_F V_2 \longrightarrow W_1 \otimes_F W_2$ is characterized by

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2).$$

Some comments are in order, however, especially in view of the caution just given. Certainly the displayed equation determines $f_1 \otimes f_2$, since elements of the form $v_1 \otimes v_2$ span $V_1 \otimes_F V_2$. But if we use the equation to *define* $f_1 \otimes f_2$, there is much to check in order to know it is well-defined. Much better is to use the universal property to define it, then check that it is in fact given by the simple formula above. To do this we need only find a suitable bilinear map $V_1 \times V_2 \longrightarrow W_1 \otimes_F W_2$. The composite

$$V_1 \times V_2 \xrightarrow{f_1 \times f_2} W_1 \times W_2 \longrightarrow W_1 \otimes_F W_2,$$

where the second map is the canonical bilinear map of the proposition, does the job. Moroever, tracing through the construction we find that it is indeed given by the displayed equation.

Here are some further properties:

Proposition 1.3 *There are natural isomorphisms of vector spaces:*

a) identity: $F \otimes_F V \cong V$.

b) associativity: $(V \otimes_F W) \otimes_F U \cong V \otimes_F (W \otimes_F U)$.

c) commutativity: $V \otimes_F W \cong W \otimes_F V$.

d) distributivity: $V \otimes (\oplus_\alpha W_\alpha) \cong \oplus_\alpha (V \otimes W_\alpha)$. *(The direct sum is over an arbitrary index set.)*

Proof: We give a brief sketch and leave the details to the reader.

a) Scalar multiplication $F \times V \longrightarrow V$ is bilinear, and so induces a linear map $\phi : F \otimes_F V \longrightarrow V$. It is clear that ϕ is an isomorphism, with inverse $v \mapsto 1 \otimes v$.

b) One approach is to first define an analogue of the tensor product for trilinear maps $V \times W \times U \longrightarrow T$. This yields an object $V \otimes_F W \otimes_F U$ with a universal property for trilinear maps. Then show that it is naturally isomorphic to each of the two vector spaces in (b).

c) This one is easy. Use the universal property again to get a map $V \otimes_F W \longrightarrow W \otimes_F V$ such that $v \otimes w \longrightarrow w \otimes v$. Symmetry yields an inverse.

d) Use the universal property of tensor products and of direct sums to define natural maps in each direction, then check they are mutually inverse.

Now suppose that V, W are representations of a group G over F. Then $V \otimes_F W$ is also a representation, with $g \cdot (v \otimes w) = gv \otimes gw$. The fact that this is a well-defined representation follows from the functoriality of the tensor product (in fact any functor between categories takes group actions to group actions; I leave it to the categorically-minded reader to decipher this statement).

Examples. 1. The simplest case is when one of the representations, say V, is 1-dimensional. Then we can assume $V = F$ with G acting via some homomorphism $\lambda : G \longrightarrow F^\times$; we denote this representation F_λ. (See below for a thorough discussion of 1-dimensional representations). Using our natural isomorphism $F \otimes_F W \cong W$, we see that $V \otimes_F W$ is just W as a vector space, but with the "twisted" G-action $g \cdot w = \lambda(g)gw$. This makes it clear that tensoring with F_λ preserves irreduciblility.

2. Tensor products of permutation representations also have a simple interpretation, in terms of products of G-sets. See the exercises.

3. In general it isn't easy to decompose the tensor product $V \otimes W$ of two irreducible representations into irreducibles, and indeed much research has been done on this topic for various groups G. But it is easy to give examples where such a tensor product can't possibly be itself irreducible. For example, when $F = \mathbb{C}$ and $G = S_3$ we saw that there is one irreducible V of dimension 2, and two more of dimension 1. So $V \otimes_{\mathbb{C}} V$ cannot be irreducible, as it has dimension 4. See the exercises for further examples.

2 Hermitian inner products

This section expands on Serre's remarks at the bottom of p. 6.

Let V, W be complex vector spaces. A map $f : V \longrightarrow W$ is *semilinear* if it is a group homomorphism and $f(zv) = \overline{z} f(v)$ for all $v \in V$, $z \in \mathbb{C}$. A *Hermitian inner product*, or simply "inner product" on V is a map $\beta : V \times V \longrightarrow \mathbb{C}$ that is linear in the first variable, semilinear in the second variable, and such that β is positive definite, i.e. $\beta(v, v) > 0$ for all $v \neq 0$. Serre calls these *scalar products*.

Often we write $(v|w)$ for $\beta(v, w)$. If $V = \mathbb{C}^n$, then

$$((a_1, ..., a_n)|(b_1, ..., b_n)) = \sum_{i=1}^{n} a_i \overline{b_i}$$

is a Hermitian inner product. In fact a routine argument shows that for any V of dimension n and inner product β on V, there is a basis $e_1, ..., e_n$ for V such that in the corresponding coordinates β is given by the displayed formula. In particular the e_i's form an orthonormal basis with respect to β.

Given an inner product β on V, the corresponding *unitary group* $U(V, \beta)$ is the group of complex linear automorphisms g of V that preserve β. In other words, $\beta(gv, gw) = \beta(v, w)$ for all v, w. If $V = \mathbb{C}^n$ with its standard inner product given above, then we write $U(n)$ for the corresponding unitary group; it is the subgroup of $GL(n, \mathbb{C})$ consisting of matrices A such that $A\overline{A}^T = Id$. By the remarks of the preceding paragraph, any $U(V, \beta)$ is isomorphic to $U(n)$, where $n = dim\, V$. (A cultural aside: $U(n)$ is a compact topological group and even a Lie group. It has many beautiful, deep topological properties.)

Returning to our finite group G, we have:

Proposition 2.1 *Let V be a complex representation of G. Then there is a G-invariant inner product β on V, i.e. such that $\beta(gv, gw) = \beta(v, w)$ for all $g \in G$.*

Proof: Choose any inner product α on V and average it:

$$\beta(v, w) = \frac{1}{|G|} \sum_{g \in G} \alpha(gv, gw).$$

It is easy to check that β is an inner product; for the positive definite part note that any linear combination of positive definite forms with positive real coefficients is again positive definite. The G-invariance is also clear.

Thus we may assume that ρ_V maps into the unitary group of some inner product, or even that $\rho_V : G \longrightarrow GL_n\mathbb{C}$ maps into $U(n)$. Such representations are called *unitary.* If V is a unitary representation of G, and $W \subset V$ is a G-invariant subspace, then the orthogonal complement $W^\perp$ is also G-invariant. This yields another proof of complete reducibility, in the special case of complex representations.

3 Representations of abelian groups; one-dimensional representations

This material appears later in Serre (see e.g §3.1). But given that these are the easiest cases, it makes sense to consider them right away.

3.1 1-dimensional representations

In this subsection we work over an arbitrary field F, and take G to be an arbitrary (not necessarily finite) group.

First of all, every one-dimensional representation is irreducible, and so up to isomorphism is given by a linear action of G on F, or equivalently a homomorphism $\rho : G \longrightarrow F^\times$. Two such homomorphisms yield isomorphic representations if and only if they are conjugate by an element of $F^\times$, and as $F^\times$ is abelian this means they are isomorphic as representations if and only if they are equal. Furthermore ρ factors uniquely through the abelianization of G. To sum up:

Proposition 3.1 *Let G be any group, F any field. Then isomorphism classes of 1-dimensional representations of G over F are in bijective correspondence with $Hom\,(G, F^\times) \cong Hom\,(G_{ab}, F^\times)$.*

Example. Take $G = S_n$. Then for all $n > 1$ the commutator subgroup is A_n, so $(S_n)_{ab} = C_2$. Hence for any field F of characteristic not 2, there are two 1-dimensional representations up to isomorphism: the trivial one, and the sign representation given by $\sigma \cdot a = (sgn\,\sigma)a$ for $\sigma \in S_n$, $a \in F$. If $char\,F = 2$, these two coincide $(1 = -1!)$ and there is only the trivial one-dimensional representation.

Now for any group G and abelian group H, $Hom\,(G, H)$ is an abelian group under pointwise multiplication: If $f_1, f_2 \in Hom\,(G, H)$, then $(f_1 \cdot f_2)(g) = f_1(g)f_2(g)$. (Check that this is an abelian group structure, and note it is essential to take H abelian.) Hence $Hom\,(G, F^\times)$ is an abelian group, and in view of the bijection of the proposition there must be some natural abelian group structure on the set $Rep^1_F(G)$ of isomorphism classes of 1-dimensional representations. What is it?

Answer: tensor product. Suppose L_1, L_2 are 1-dimensional representations of G. Then $L_1 \otimes_F L_2$ is again a 1-dimensional representation.

Proposition 3.2 *The tensor product gives $Rep^1_F(G)$ an abelian group structure, with identity element the trivial representation and with inverses given by the dual representations L^*. Moreover, the bijection of the preceding proposition is an isomorphism of groups.*

Proof: The functoriality of tensor product shows the the operation $L_1 \otimes_F L_2$ is well-defined on isomorphism classes. It is associative and commutative with identity the trivial representation by the general such rules for tensor products. Finally the evaluation map $L^* \otimes_F L \longrightarrow F$ given by $\varepsilon(\lambda \otimes v) = \lambda(v)$ is an isomorphism of representations. To see this, note that it is a nonzero F-linear map between 1-dimensional vector spaces, so is certainly an isomorphism as vector spaces. So it remains to check that ε is G-equivariant; this follows because of the inverse that appears in the definition of the dual: $(g \cdot \lambda)(v) = \lambda(g^{-1}v)$. Thus

$$\varepsilon(g \cdot (\lambda \otimes v)) = \varepsilon(g \cdot \lambda \otimes gv) = \lambda(g^{-1}gv) = \lambda(v) = g \cdot (\lambda(v)).$$

Finally, note that the natural isomorphism $F \otimes_F F \xrightarrow{\cong} F$ is just multiplication: $a \otimes b \mapsto ab$. It follows that the tensor product of L_1, L_2 corresponding to homomorphisms $f_1, f_2 : G \longrightarrow F^\times$ is isomorphic to the representation L corresponding to $f_1 f_2$. This proves the last statement of the proposition.

Remark. Note that we now have an action of the group $Rep^1_F G$ on the set $Rep^n_F G$ for all n, given by tensor product. This helpful when constructing character tables; see below.

3.2 Representations of abelian groups over $\mathbb{C}$

In this section we could take F to be any algebraically closed field of characteristic zero, but to avoid distractions we'll take $F = \mathbb{C}$.

Proposition 3.3 *Let G be any abelian group (not necessarily finite). Then every irreducible representation of G over $\mathbb{C}$ is 1-dimensional.*

Proof: Let $\rho : G \longrightarrow GL_n\mathbb{C}$ be a representation. Then $Im\, \rho$ is a commuting set of linear transformations and hence by a previous exercise can be simultaneously triangularized. In particular, the elements of $Im\, \rho$ have a common eigenvector, i.e. there is a line L invariant under $Im \rho$ and hence invariant under the G-action. So if the representation is irreducible then $n = 1$, QED.

Now return to our assumption G finite. The next lemma was stated without proof in exercise 3 of the notes on algebras over a field.

Lemma 3.4 *Let G be a finite abelian group. Then $Hom\,(G, \mathbb{C}^\times)$ is (non-naturally) isomorphic to G.*

Proof sketch (you fill in the details): By the classification theorem for finite abelian groups, G is a direct product of cyclic groups (note this involves some arbitary choices). This reduces the problem to the case of cyclic groups. That case can be done using the fact that the torsion subgroup of $C^\times$ is the union of the cyclic subgroups μ_n (n-th roots of unity), although again some arbitrary choices must be made.

We then conclude:

Proposition 3.5 *For any finite group G, the group of 1-dimensional representations $Rep^1_{\mathbb{C}}G$ is naturally isomorphic to $Hom\,(G_{ab}, \mathbb{C}^\times)$, and hence (non-naturally) isomorphic to G_{ab}. In particular, there are $|G_{ab}|$ 1-dimensional representations, up to isomorphism.*

Example. Both the quaternion group Q_8 and the dihedral group D_8 have abelianization $C_2 \times C_2$ (check this!). Hence they have four one-dimensional representations. Taking Q_8 to illustrate, these representations are given as composite homomorphisms

$$Q_8 \longrightarrow C_2 \times C_2 \xrightarrow{\lambda} C_2,$$

where λ ranges over the four elements of $Hom\,(C_2 \times C_2, C_2)$.

4 Serre, Chapter 2

Some results can be simplified significantly by making use of the group algebra. For example, Proposition 6 and Theorem 6 look more complicated than they really are. I'll give simplified proofs in class.

When classifying irreducible representations and constructing character tables, whenever possible you should incorporate the action of $Rep^1_F G$ on $Rep^n_F G$. See for instance the S_4 example in §5.8 of Serre. Our magic formulas reveal that S_4 has two distinct 3-dimensional irreducibles, one of which is the standard $(n-1)$-dimensional representation W of S_n you constructed in an exercise. But what's the other one? Your first guess should be to tensor W with the sign representation (denoted ε in Serre). It is possible that this doesn't change the isomorphism type of W; for example, in the S_3 analogue it does not. Here you can read off immediately from the characters that W and $\mathbb{C}_{sgn} \otimes_{\mathbb{C}} W$ have different characters and hence are not isomorphic.

Proposition 7 I call the Second Orthogonality Formula. Here's where it comes from: A square matrix has orthonormal rows if and only if it has orthonormal columns. The first orthogonality formula (Theorem 3) can be viewed as expressing orthogonality of the rows of a certain matrix. Orthogonality of the columns then yields Proposition 7!

To be continued in Part II.

5 Exercises

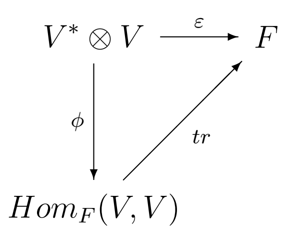
1. Suppose V, W are finite dimensional vector spaces over a field F.
   a) Show that there is a natural isomorphism of vector spaces

$$\phi : V^* \otimes W \xrightarrow{\cong} Hom_F(V, W).$$

Your isomorphism should be "natural" enough to show simultaneously: If V, W are FG-modules, then it is an isomorphism of FG-modules, as well as to prove part (b) below.

Note. The dimensions of the source and target are the same, so of course *some* isomorphism as vector spaces exists. But there is only one correct, natural definition of ϕ.

b) Let $\varepsilon : V^* \otimes V \longrightarrow F$ denote the evaluation map, given by $\varepsilon(\lambda \otimes v) = \lambda(v)$. Show that the isomorphism ϕ constructed in (a) is such that the following diagram commutes:



This gives an elegant, coordinate-free interpretation of the trace. (If your ϕ doesn't have this property, go back to the drawing board!)

2. *Tensor products of algebras.* Suppose R, S are F-algebras. Show that the tensor product $R \otimes_F S$ has an F-algebra structure such that $(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = (r_1 r_2 \otimes s_1 s_2)$. The structure map (required for F-algebras) $F \longrightarrow R \otimes_F S$ is given by $a \mapsto a \otimes 1 = 1 \otimes a$. Now prove the following isomorphisms of F-algebras:

a) $F[x_1, ..., x_m] \otimes_F F[y_1, ..., y_n] \cong F[x_1, ...x_m, y_1, ..., y_n]$.

b) For arbitrary groups G, H (not necessarily finite), $FG \otimes_F FH \cong F(G \times H)$.

c) For any F-algebra A, $M_n F \otimes_F A \cong M_n A$. Conclude from this that $M_m F \otimes_F M_n F \cong M_{mn} F$.

d) Optional problem: Show that for commutative F-algebras R, S, the tensor product $R \otimes_F S$ is the coproduct in the category of commutative F-algebras.

3. *Permutation representations.* Consider the functor Φ : **G-set** $\rightarrow$ **F-mod** that takes a G-set X to FX with the evident G-action: G acts on the given basis X of FX using the given action on X.

a) Given two G-sets X, Y, we can form their disjoint union $X \coprod Y$ and their Cartesian product $X \times Y$. Show that Φ takes disjoint unions to direct sums, and Cartesian products to tensor products. More precisely, show that there is a natural isomorphism

$$F(X \times Y) \cong FX \otimes_F FY.$$

and similarly for disjoint unions/direct sums. (As usual, you're not required to prove naturality in the technical categorical sense of the term; just be sure your isomorphisms don't depend on arbitrary choices.)

b) What is the dimension of $(FX)^G$? Look for a simple answer involving only the G-set X.

c) Take $F = \mathbb{C}$, $G = S_4$ and $X = \mathcal{P}_2[4]$ (the set of 2-element subsets of [4]), with S_4 acting in the evident way. Compute the character of $\mathbb{C}X$ and, using characters only, determine the composition factors (=irreducible summands in our completely reducible setting) with multiplicities. (Use the character table for S_4 given in Serre §5.8.)

4. The standard 2-dimensional irreducible representation of D_8 over $\mathbb{R}$ can be regarded as a representation over $\mathbb{C}$, and as such it is still irreducible. Call this complex representation V.

a) Write down the character table for D_8. (This is discussed in Chapter 5 of Serre. Note his D_n is my D_{2n}, as I am indexing these groups by their order.)

b) Use Theorem 4 of Serre to determine the decomposition of $V \otimes_{\mathbb{C}} V$ into irreducibles. (In other words, use characters only.)

c) Using the standard basis e_1, e_2 for V, exhibit explicitly a decomposition as in (b).

5. Let p be an odd prime, and let $G = Aff_1\mathbb{F}_p$. Recall that this is the semidirect product $T \rtimes \mathbb{F}_p^\times$, where T is the group of translations of $\mathbb{F}_p$. Thus $T \cong \mathbb{Z}/p$ (beware the mixing of additive and multiplicative notation), and under this isomorphism the action of $\mathbb{F}_p^\times$ is just the usual action by multiplication on $\mathbb{Z}/p = \mathbb{F}_p$. All questions refer to representations over $\mathbb{C}$.

a) How many irreducible representations does G have (up to isomorphism, of course).

b) What are the dimensions of the irreducibles?

c) Give explicit constructions of the irreducibles.

d) For $p = 5$, give the complete character table. (As a check on your work, make sure the orthogonality relations are satisfied.)

Note. If it helps, you might want to note that the case $p = 3$ is just our old friend S_3.