The Duflot filtration in equivariant topology

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1 Introduction

In a landmark two-part paper [], Daniel Quillen began a new era in the study of mod p cohomology rings of finite groups. He determined many commutative algebra invariants of H^*_G in terms of the group structure of G alone: the Krull dimension of H^*_G is the maximal rank of a p-torus (i.e, elementary abelian p-subgroup) in G, the number of minimal primes is the number of conjugacy classes of maximal p-tori, and the prime ideal spectrum has a stratification determined in an explicit way by the p-tori. One of the main tools used was to choose an embedding of G in a unitary group U and a maximal torus in U. Letting S denote the corresponding maximal p-torus, we get a smooth action of G on U/S and a fibration $EG \times_G (U/S) \longrightarrow BG$. The induced map on cohomology is faithfully flat, so one can try to prove the desired results first for $H^*_G(U/S) := H^*(EG \times_G (U/S)$, and then apply faithfully flat descent to deduce the results for $H^*_G = H^*BG$. It turns out that this works beautifully.

Another essential feature of Quillen's method was that rings were considered up to F-isomorphism. An F-isomorphism is a ring homomorphism $\phi : R \longrightarrow R'$ such that the kernel of ϕ consists of nilpotent elements, and for all $a \in R'$ some p^k-th power of a is in the image of ϕ. As a result, the method has an inherent limitation: it cannot detect commutative algebra invariants that aren't preserved by F-isomorphism, such as depth, associated primes, local cohomology, and of course the nilradical itself. New methods are needed.

The first steps in this direction were taken by Quillen's student Jean Duflot [refs]. Duflot proved that the depth of H^*_G is at least the rank of a maximal central p-torus in G, and that the associated primes are all p-toral, meaning that if $\mathfrak{p}$ is an associated prime then $\mathfrak{p}$ is essentially the kernel of restriction to a p-torus. In addition to descent, Duflot made use of two new ingredients. First is the observation that $H^*_G(U/S) \cong H^*_S(G\backslash U)$. We call this the *exchange isomorphism*. Second is a beautiful theorem of independent interest on the equivariant cohomology of smooth S-manifolds. If M is a smooth S-manifold, it has a filtration by ranks of isotropy groups. The surprising fact, proved by Duflot, is that the filtration is very faithfully reflected in equivariant cohomology, and the quotients of the cohomology filtration are direct sums of modules of a particularly simple form. This facilitates analysis of $H^*_S M$, in particular for $M = G\backslash U$. The results obtained are then transferred back to H^*_G by exchange and descent.

Grothendieck's local cohomology (introduced below in the appendix) is another important invariant. In particular, one can distill from it an integer-valued invariant called Castenuovo-

Mumford regularity, or simply "regularity". David Benson conjectured that the regularity $reg\, H^*_G = 0$, and showed that $reg\, H^*_G \geq 0$. In 2010 Peter Symonds used Duflot's methods to show that $reg\, H^*_G \leq 0$, and the conjecture was proved. With the advantage of hindsight, and without in any way detracting from Symonds beautiful theorem, it is fair to say that the inequality $reg\, H^*_G \leq 0$ is an almost immediate consequence of Duflot's methods. In principle, it could have been proved two decades earlier. In a nutshell: The direct summands occuring in the quotients of the Duflot filtration for the S-manifold $G\backslash U$ are easily seen to have regularity at most $dim\, U$. Hence the same is true for $H^*_S(G\backslash U)$, and then by the exchange isomorphism the same is true for $H^*_G(U/S)$. By descent this immediately implies $reg\, H^*_G \leq 0$.

Perhaps one lesson to be drawn from Symonds' proof is that Duflot's method still has not been fully exploited. This thought lead the authors to revisit the Duflot filtration and to consider systematically its applications to group cohomology. Our main purpose here is to give an exposition of the Duflot method and its applications. Most of the results are not new, although in some cases the proofs may be new. Any new results are taken from the first author's thesis. For example, the "Duflot complex", although an immediate consequence of the Duflot filtration, does not seem to have appeared in print before. One major set of results from the thesis will not be considered here: an axiomatization of the key features of the Duflot filtration, in the context of general commutative algebras.

In any case, we wish to emphasize that the main ideas behind our paper are due to Duflot, with many further ideas and/or results taken from Symonds, Jon Carlson, Dave Benson, Burt Totaro, Nick Kuhn, and others. Most of the results we discuss apply with suitable modifications to general compact Lie groups, but to simplifty the exposition we will confine our attention to finite groups.

Outline of the paper.

2 Descent

In the introduction we alluded to the method of faithfully flat descent. For the most part, we won't use the full strength of this method; we simply exploit directly the fact that $H^*_G(U/S)$ is a finitely-generated free module over H^*_G.

2.1 Descent for $EG \times_G (U/S) \longrightarrow BG$

Choose complex representations $V_1, ..., V_m$ of G such that the representation on $\oplus V_i$ is faithful. Let $U = \prod_{i=1}^m U(V_i) \subset U(V)$ denote the corresponding product of unitary groups. Let $T \subset U$ be a maximal torus compatible with the product decomposition, and let $S = \{u \in T : u^p = 1\}$—a p-torus of rank $n = \sum n_i$. Then the homogeneous space U/S is a smooth principal T/S-bundle over the product of complete flag varieties U/T. We obtain smooth, compatible actions of G on U/T and U/S. Two key properties of the action on U/S are as follows:

Proposition 2.1 *The isotropy groups of the G-action on U/S are all p-tori. Moreover, for every p-torus $A \subset G$, $(U/S)^A$ is nonempty.*

This is clear. The second property is:

Proposition 2.2 *The Serre spectral sequence of $EG \times_G (U/S) \longrightarrow BG$ collapses. Hence $H^*_G(U/S)$ is a finitely-generated free module over H^*_G, non-naturally isomorphic to $H^*_G \otimes H^*(U/S)$.*

For the proof it is enough to show collapse in the universal case $EU \times_U (U/S) = BS \longrightarrow BU$. This is an easy exercise, because the cohomology ring Poincaré series of BU, BS, U/S and so on are all explicitly known and one can force the collapse by a dimension-counting argument.

In particular, $H^*_G \longrightarrow H^*_G(U/S)$ is finite and faithfully flat. As a result, many commutative algebra properties of H^*_G are equivalent to, or at least can be deduced from, corresponding properties of $H^*_G(U/S)$. For example:

1. H^*_G and $H^*_G(U/S)$ have the same Krull dimension.
2. H^*_G and $H^*_G(U/S)$ have the same depth.
3. The local cohomologies are related by a non-natural isomorphism
$$\mathfrak{h}^i H^*_G \otimes H^*(U/S) \cong \mathfrak{h}^i H^*_G(U/S).$$
4. The Castelnuovo-Mumford regularity satisfies $reg\, H^*_G + dim\,(U/S) = reg\, H^*_G(U/S)$.
5. The induced map $Spec\, H^*_G(U/S) \longrightarrow Spec\, H^*_G$ induces a surjection on associated primes. ([Matsumura], 9B Corollary).

Remark. Often we will require that the V_i's are irreducible, so that the center of G is contained in the center of U. In particular, the maximal central p-torus C of G then acts trivially on U/S.

2.2 Descent for fixed-points and centralizers

Let $A \subset G$ be a p-torus of rank i, and let X be a path-component of $(U/S)^A$. Then as a representation of A, V has a functorial decomposition into isotypical summands: $V = \oplus_{s=1}^t W_s$. Let X be a path-component of $(U/S)^A$. Then one can easily check the following:

1. $C_U A = \prod U(W_s)$ is itself a product of unitary groups, and in particular is connected.
2. $C_U A$ acts transitively on X, with isotropy groups conjugates of S that are compatible with the V_i's and with the isotypical decomposition. Thus X is a T/S-bundle over a product of complete flag manifolds.
3. The following natural maps are all diffeomorphisms:
$$U \times_{C_U A} X \xrightarrow{\cong} U \qquad G \times_{C_G A} X \xrightarrow{\cong} GX \qquad N_G A \times_{C_G A} X \xrightarrow{\cong} (N_G A)X.$$

In short, if X is a component of the A-fixed points of G-manifold of type U/S, then X is itself a $C_G U$-manifold of type U/S. In particular the results of the preceeding section apply to it.

Remarks. 1. The components of $(U/S)^A$ are all diffeomorphic, and in particular of the same dimension. Note also that if $\pi : U/S \longrightarrow U/T$ is the natural map, then $(U/S)^A = \pi^{-1}((U/T)^A)$, so that fixed points can be determined by looking at the complete flag manifold or product of complete flag manifolds.

2. If X is a component of $(U/S)^A$, let $X_A = \{x \in X : G_x = A\}$. Then X_A is open in X and connected, and if it is nonempty it is dense in X. The noteworthy point here is the connectivity. The complement of X_A in X is a finite union of submanifolds of lower dimension, namely the fixed-point components of various p-tori containing A. But these fixed-point components are again T/S-bundles over products of complete flag manifolds, from which it follows easily that the submanifolds in question have codimension at least 2. This in turn implies that X_A is connected.

3 Exchange

3.1 The exchange isomorphism for U/S and $G\backslash U$

Suppose H, K are topological groups, and W is a space with a free left H-action and a free right K-action, such that the two actions commute and the projections $W \longrightarrow H\backslash W$ and $W \longrightarrow W/K$ are principal bundles.

Proposition 3.1 *There is a natural weak equivalence $EH \times_H (W/K) \cong (H\backslash W) \times_K EK$.*

Proof: Consider $EH \times_H W \times_K EK$. Since K acts freely on W, the natural map $EK \times_K W \longrightarrow W/K$ is a fibration with contractible fiber, hence a homotopy equivalence. Applying the functor $EH \times_H (-)$ to this map yields an equivalence

$$EH \times_H W \times_K EK = EH \times_H (W \times_K EK) \longrightarrow EH \times_H (W/K).$$

Similarly, $EH \times_H W \times_K EK \cong (H\backslash W) \times_K EK$.

Corollary 3.2 $H^*_H(W/K) \cong H^*_K(H\backslash W)$.

Applying this to the case $H = G$, $K = S$, $W = U$, we obtain an isomorphism of algebras

$$H^*_G(U/S) \cong H^*_S(G\backslash U).$$

We call this the *exchange* isomorphism. It exchanges an action of a complicated group G on a simple space U/S for an action of a p-torus S on a more complicated space $G\backslash U$.

The general exchange isomorphism has the following naturality property. Suppose $H_0 \subset H$, $K_0 \subset K$ are subgroups, and $W_0 \subset W$ is a subspace invariant under H_0, K_0 and such that the principal bundle condition is satisfied. Then there is a commutative square

$$
\begin{array}{ccc}
H^*_H(W/K) & \xrightarrow{\cong} & H^*_K(H\backslash W) \\
\downarrow & & \downarrow \\
H^*_{H_0}(W_0/K_0) & \xrightarrow[\cong]{} & H^*_{K_0}(H_0\backslash W_0)
\end{array}
$$

where the horizontal maps are exchange isomorphisms and the entire diagram is induced by an evident commutative diagram of spaces.

3.2 The exchange isomorphism for fixed-point sets and centralizers

Let $A \subset G$ be a p-torus, X a component of $(U/S)^A$, and choose $x \in X$. We have $xS \in (U/S)^A$ if and only if $x^{-1}Ax \subset S$. Let $Bx^{-1}Ax$. Note that since X is connected, B is independent of the choice of x. Then $\{u \in U : u^{-1}Au = B\}$ is a compact $(G \times S)$-submanifold. Let Z denote a component of this submanifold. Then $X := Z/S$ is a component of $(U/S)^A$ and $Y := G\backslash(GZ) = C_GA\backslash Z$ is a component of $(G\backslash U)^B$. Recalling that $H^*_G(G \times_{C_GA} X) = H^*_{C_GA}X$, we then have exchange on the level of fixed-point components in the form

$$H^*_{C_GA}X \cong H^*_SY.$$

3.3 Duflot's theorem on depth

The following theorem is due to Duflot. Recall that c_pG denote the rank of a maximal central p-torus C_p of G.

Theorem 3.3 *depth* $H^*_G \geq c_pG$.

Proof: We first show that depth is preserved under descent. The depth of H^*_G is the smallest i such that the local cohomology $\mathfrak{h}^iH^*_G$ is nonzero, and similarly for $H^*_G(U/S)$. (For us, the local cohomology of a connected graded algebra is always at the ideal $\mathfrak{m}$ of positive dimensional elements; hence $\mathfrak{m}$ is omitted from the notation. See the appendix.) Since $H^*_G \longrightarrow H^*_G(U/S)$ is a finite algebra homomorphism, by a corollary of the Independence Theorem $\mathfrak{h}^*H^*_G(U/S) = \mathfrak{h}^*_{H^*_G}H^*_G(U/S)$. But $H^*_G(U/S)$ is a free H^*_G-module, so *depth* $H^*_G =$ *depth* $H^*_G(U/S)$ as claimed.

Depth is clearly preserved under exchange, since $H^*_G(U/S) \cong H^*_S(G\backslash U)$ as rings. Next observe the general fact:

Lemma 3.4 *If M is a smooth S-manifold, and K is the kernel of the action, then depth $H^*_SM \geq$ rank K. In fact*

$$\mathit{depth}\, H^*_SM = \mathit{rank}\, K + \mathit{depth}\, H^*_{S/K}M.$$

Proof: There is a natural isomorphism $H^*_SM \cong H^*_K \otimes H^*_{S/K}M$. The lemma then follows from the Kunneth theorem in local cohomology.

By our convention on the choice of representation $G \subset U$, we have $C_p \subset C_p(U) \subset S$. Then C_p acts trivially on U/S and on $G\backslash U$, so $depth\, H^*_G \geq rank\, K \geq c_p$. This completes the proof of Duflot's theorem.

Example. It is shown in [Carlson-Henn] that depth for wreath products satisfies $depth\,(H \int \mathbb{Z}/p) = depth\, H + 1$. In particular, $depth\,(\int^n \mathbb{Z}/p) = n$.
But the center of $(\int^n \mathbb{Z}/p)$ is just $\mathbb{Z}/p$ for all n. This shows that the depth can exceed the Duflot bound by an arbitrarily large number.

Example. The semidihedral group of order 16 has $c = 1$ and $depth = 1$, so the Duflot bound is sharp in this case. But as far as we know, the only way to compute the depth here is by explicitly calculating the cohomology ring. [reference]

3.4 Exchange notation and conventions

Notation: The following notation will be fixed throughout the remainder of the paper:

1. A is a p-torus in G and X is a component of $(U/S)^A$.

2. B is the corresponding p-torus in S and Y the corresponding component of $(G\backslash U)^B$. ("Corresponding" according to the recipe of §3.2.)

Note that the roles of A, X and B, Y could be reversed here; we could just as well start with B, Y and obtain the corresponding A, X.

Conventions: The S-manifold $G\backslash U$ has a number of special properties not shared by general S-manifolds. We point out in particular the following:

1. If $B \subset S$ and Y is a component of $(G\backslash U)$, then Y_B is connected (if it is nonempty, its complement has codimension 2, as discussed earlier).

2. With the notation of the previous item, S preserves each such component Y. In a general S-manifold M, S would permute the components, but here the S-action is restricted from a T-action that also permutes the components, and since T is connected the permutation is trivial.

Purely as a matter of convenience, we make the convention that all of our S-manifolds have these two properties. This simplifies the notation below significantly.

4 The Duflot filtration

4.1 An example and an outline

Let S be the group of p-th roots of unity, p odd. Regard S^3 as the unit sphere in $\mathbb{C}^2$, and let S act on S^3 by $\xi \cdot (z_1, z_2) = (\xi z_1, z_2)$. The fixed-point set is the circle defined by $z_1 = 0$. The local coefficient system of the fibration $S^3 \longrightarrow ES \times_S S^3 \longrightarrow BG$ is trivial, so the E_2-term of the Serre spectral sequence is $H^*BS \otimes H^*S^3$. Since the fibration has a section given by choosing a fixed point (in fact this section is unique up to fiber homotopy, since the fixed-point set is connected) there can be no differentials; the spectral sequence collapses. Since p is odd there are no multiplicative extension issues in this case, and we conclude that as algebras

$$H^*_S S^3 \cong H^*_S \otimes H^*S^3.$$

But there is another way of analyzing $H^*_S S^3$, by stratifying S^3 by p-ranks of isotropy groups. In this case there are just two strata: the closed set of rank 1 points, i.e. our fixed-point set S^1, and the open set $S^3 - S^1$ of free points. We will examine the long exact sequence in equivariant cohomology

$$\ldots \longrightarrow H^*_S(S^3, S^3 - S^1) \longrightarrow H^*_S S^3 \longrightarrow H^*_S(S^3 - S^1) \longrightarrow \ldots$$

Noting that S acts on $\mathbb{C}P^1$ in a similar way, so that the natural map $\pi : S^3 \longrightarrow \mathbb{C}P^1$ is S-equivariant, we see that $X - S^1$ is the inverse image of the complement of a point. It follows easily that $X - S^1$ is S-diffeomorphic to $S^1 \times \mathbb{C}$, with S acting in the usual way on S^1 and trivially on $\mathbb{C}$. Since the action is free, we obtain

$$H^*_S(S^3 - S^1) \cong H^*((S^3 - S^1)/S) \cong H^*(S^1).$$

Now let $\mathcal{E}$ denote the total space of the normal bundle to S^1 in S^3, which is a S-vector bundle. By excision $H^*_S(S^3, S^3 - S^1) \cong H^*_S(\mathcal{E}, \mathcal{E} - S^1)$, so we are in a position to apply the Thom isomorphism. For this, however, we need to know that the normal bundle is orientable. But the normal bundle is pulled back from the tangent space of $[0,1] \in \mathbb{C}P^1$, so $\mathcal{E}$ in fact has a complex structure and hence the bundle $\nu = ES \times_S \mathcal{E} \longrightarrow ES \times_S S^1 = BS \times S^1$ has a complex structure. Moreover the Euler class $e(\nu) = z \otimes 1 \in H^*BS \times S^1$ and hence is a non-zerodivisor. Using the general fact that the Thom isomorphism followed by restriction to the zero section gives multiplication by the Euler class, we find that the above long exact sequence reduces to a short exact sequence

$$0 \longrightarrow H^*_S(S^3, S^3 - S^1) \longrightarrow H^*_S S^3 \longrightarrow H^*_S(S^3 - S^1) \longrightarrow 0,$$

or

$$0 \longrightarrow \Sigma^2(H^*(BS \times S^1)) \longrightarrow H^*_S S^3 \longrightarrow H^*S^1 \longrightarrow 0.$$

4.2 Equivariant characteristic classes for certain $G \times H$-spaces

This section contains a preliminary result needed for the Duflot filtration.

Let X be a $(G \times H)$-space such that H acts trivially and G acts freely on X. Then

$$(EG \times EH) \times_{G \times H} X = EG \times_G (BH \times X) \cong (BH \times X)/G = BH \times (X/G).$$

In particular we have:

Proposition 4.1 $H^*_{G\times H}X \cong H^*BH \otimes H^*(X/G)$.

Now consider an equivariant real vector bundle $E \downarrow X$. We do not assume H acts trivially on E. Let $W_1, ..., W_r$ denote the irreducible representations of H over $\mathbb{R}$. Then for any representation V of H we have the isotypical decomposition (cf. Brocker-tom Dieck, p. 70)

$$\oplus_{i=1}^r Hom_H(W_i, V) \otimes_{\mathbb{R}} W_i \xrightarrow{\cong} V.$$

This decomposition is natural in V and consequently extends to vector bundles, where now W_i is regarded as the H-vector bundle $X \times W_i$. Since the G and H actions commute, the action of G preserves the isotypical summands and the tensor product decomposition of the summands. In particular $Hom_H(W_i, V)$ is a G-vector bundle over X, and since the action is free we know that it is pulled back from an ordinary vector bundle V_i over X/G (cf. Atiyah). Summing up, we have:

Proposition 4.2 *As $(G \times H)$-vector bundles there is a natural decomposition*

$$E \cong \oplus_{i=1}^r W_i \otimes \pi^* V_i$$

where the W_i's are the irreducible representations of H, G acts trivially on W_i, $\pi : X \longrightarrow X/G$ is the quotient map and $V_i \downarrow X/G$ is a vector bundle on the orbit space (so π^*V_i is a G-vector bundle over X).

We next observe that if E is a complex vector bundle, F a real vector bundle, then $E \otimes_{\mathbb{R}} F$ receives a complex structure from E, and there is a natural isomorphism of complex vector bundles

$$E \otimes_{\mathbb{R}} F = E \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} F).$$

In particular, when E is a complex line bundle and $dim\, F = n$, setting $z = c_1 E$ we get

$$e(E \otimes_{\mathbb{R}} F) = c_n(E \otimes_{\mathbb{C}} F_{\mathbb{C}}) = z^n + c_1(F_{\mathbb{C}})z^{n-1} + ... + c_n(F_{\mathbb{C}}).$$

Note that if we take mod p coefficients with p odd, the Chern classes of $F_{\mathbb{C}}$ are the Pontrjagin classes of F (up to sign, if one uses Milnor's convention for Pontrjagin classes).

We next apply this observation to equivariant characteristic classes. Note that any oriented real 2-plane bundle has a canonical complex structure; similarly, any oriented 2-dimensional representation of a compact Lie group has a canonical complex structure. If H is a finite abelian group of odd order (or a torus), then every non-trivial irreducible real representation W has dimension 2. In the next proposition the notation is as in Proposition 4.2, and coefficients are in $\mathbb{F}_p$.

Proposition 4.3 *Suppose p is odd and H is a non-trivial finite abelian p-group, or a torus, and W_i is non-trivial. Then $W_i \otimes \pi^* V_i$ is complex (in particular oriented), and if $z \in H^2BH$ is the equivariant c_1 of W_i, and $n = dim\, V_i$, then in the notation of Proposition 4.1 the $(G \times H)$-equivariant Euler class is given by*

$$e(W_i \otimes \pi^* V_i) = z^n \otimes 1 + z^{n-1} \otimes c_1 \otimes (\pi^*(V_i)_{\mathbb{C}}) + ...1 \otimes c_n(\pi^*(V_i)_{\mathbb{C}}).$$

In particular this Euler class is a nonzerodivisor in $H^*_{G\times H}X$*, and if all of the* W_i*'s are non-trivial then* $e(E)$ *is a nonzerodivisor.*

For the statement about nonzerodivisors, it is only necessary to observe that z is a nonzerodivisor in H^*BH.

For the analogous statement for abelian 2-groups, we of course take coefficients $\mathbb{F}_2$ and use Stiefel-Whitney classes. But we also need to restrict to elementary abelian 2-groups, since otherwise the w_1's occuring could have square zero.

Proposition 4.4 *Suppose* H *is an elementary abelian 2-group, and* W_i *is non-trivial (and 1-dimensional, since irreducible). Then if* $z = w_1(W_i) \in H^1H$*,*

$$e(W_i \otimes \pi^* V_i) = z^n \otimes 1 + z^{n-1} \otimes w_1 \otimes (\pi^*(V_i)_{\mathbb{C}}) + ...1 \otimes w_n(\pi^*(V_i)_{\mathbb{C}}).$$

In particular this Euler class is a nonzerodivisor in $H^*_{G\times H}X$*, and if all of the* W_i*'s are non-trivial then* $e(E)$ *is a nonzerodivisor.*

5 The Duflot filtration

Note: don't want M compact, but do want finitely many fixed-point components. Best assumption?

Throughout this section, S is a p-torus of rank n and M is a smooth S-manifold. The *rank* $rank\, x$ of a point $x \in M$ is the rank of its isotropy group. We set

$$M_i^j = \{x \in M : i \leq rank\, x \leq j\}.$$

Thus M_0^j is the set of points with rank $\leq j$, while M_j^j is the set of points of rank exactly j. Then

$$M_0^0 \subset M_0^1 \subset ... \subset M_0^n$$

is an increasing filtration of M by open submanifolds. We obtain a corresponding filtration on $H^*_S M$

$$0 = \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset ... \subset \mathcal{F}_0 = H^*_S M$$

by taking

$$\mathcal{F}_i = Ker\,(H^*_S M \longrightarrow H^*_S M_0^{i-1}).$$

Of course, the true length of the filtration may be shorter. Setting $r = max\,\{rank\, x : x \in M\}$, and c equal to the rank of the kernel of the action, we can and usually will rewrite it as

$$0 = \mathcal{F}_r \subset \mathcal{F}_{r-1} \subset ... \subset \mathcal{F}_c = H^*_S M.$$

5.1 The localization tower and the main theorem

Since the M_0^j's are open submanifolds, and M_j^j is a closed submanifold of M_0^j with complement M_0^{j-1}, there is a localization sequence in equivariant cohomology

$$...\longrightarrow H_S^m(M_0^j|M_j^j)\longrightarrow H_S^m M_0^j \longrightarrow H_S^m M_0^{j-1}\longrightarrow ...$$

Here $H_S^*(M_0^j|M_j^j)$ is just $H_S^*(M_0^j, M_0^j - M_j^j)$; we think of it as "cohomology localized at the submanifold M_j^j". There are natural isomorphisms

$$H_S^*(M_0^j|M_j^j) \cong \oplus_W H_S^*(M_0^j|W)$$

where W ranges over the path-components of M_j^j. Moreover, by a combination of excision and the Thom isomorphism for the normal bundle of W, we have isomorphisms

$$H_S^*(M_0^j|W) \cong \Sigma^{cd(W)} H_S^* W.$$

We can assemble these exact sequences into an exact couple, which we display as a tower

...tower here...

However, the tower breaks up into short exact sequences:

Proposition 5.1 *The pushforward maps $H_S^*(M_0^j|M_j^j)\longrightarrow H_S^* M_0^j$ are injective.*

Proof: Consider two path-components W_1, W_2 of M_j^j. Thus W_i is a path-component of $(M_0^j)^{B_i}$ for $i = 1, 2$, where $B_i \subset S$ has rank j. If $W_1 \neq W_2$, then pushforward followed by restriction

$$H_S^{*-cd(W_1)} W_1 \cong H_S^*(M_0^j|W_1)\longrightarrow H_S^*(M_0^j)\longrightarrow H_S^* W_2)$$

is the zero map. If $W_1 = W_2$ we get multiplication by the Euler class $e(\nu_{W_1})$, which is a nonzerodivisor by ?. Hence the pushforward followed by restriction

$$\oplus_W H_S^{*-cd(W)} W \cong \oplus_W H_S^*(M_0^j|W)\longrightarrow \oplus_W H_S^* W$$

is just $\oplus_W \cdot e(\nu_W)$ and so is injective.

Corollary 5.2 *The restriction maps $H_S^* M\longrightarrow H_S^* M_0^j$ are surjective.*

Note that each W occuring in the decomposition if a component of $(M_0^j)^B$ for a unique rank j p-torus B, namely the common isotropy group of the points of W. With this notation we have the main theorem:

Theorem 5.3

$$\mathcal{F}_j/\mathcal{F}_{j+1} \cong \oplus_W \Sigma^{cd(W)} H_S^* W)$$

as $H_S^* M$-modules, where W ranges over the path-components of M_j^j.

Moreover, as algebras $H_S^* W \cong H_B^* \otimes H^*(W/S)$, with W/S a smooth manifold.

Proof: The first statement is immediate from the preceeding proposition, except perhaps for the module statement. But this follows from the fact that the pushforward $\Sigma^{cd(W)}H_S^*W \longrightarrow H_S^*M_0^j$ is a map of $H_S^*M_0^j$-modules.

For the second statement, choose a complement B' to B, so that $S = B \times B'$. Then B' acts freely on W and $W/S = W/B'$, so W/S is a smooth manifold. Moreover

$$ES \times_S W = BB \times EB' \times_{B'} W \cong BB \times (W/B'),$$

and the Kunneth theorem completes the proof.

Remark. If $p = 2$, then the theorem exhibits H_S^*W as the tensor product of a polynomial algebra and a finite-dimensional algebra. If p is odd, it is still true that H_S^*W is the tensor product of a polynomial algebra and a finite dimensional algebra: one only has to shift the exterior part of H_B^* into the second factor of the tensor product.

5.2 Another description of the filtration

Suppose B has rank s and Y is a component of M^B. Then (recall that we are assuming, for convenience, that all such Y are invariant under the S-action) Y has its own Duflot filtration. Moreover, the Duflot filtration of Y is just the intersection of Y with the Duflot filtration of M. It is then easy to show:

Proposition 5.4 *The pushforward $i_* : \Sigma^{cd(Y)}H_S^*Y \longrightarrow H_S^*M$ associated to $i : Y \subset M$ is injective, preserves Duflot filtrations, and is compatible with the canonical direct sum decompositions of the layers.*

The fact that i_* preserves Duflot filtrations follows from naturality of the pushforward. The compatibility with the direct sum decompositions follows from functoriality of the pushforward. More precisely, suppose $B \subset B'$ and Y' is a component of $Y^{B'}$ with $W := Y_{B'} \neq \emptyset$. Then the summand H_S^*W occuring in the appropriate Duflot layer of H_S^*Y maps by the identity to the corresponding summand of the corresponding Duflot layer of H_S^*M. Thus i_* is injective on each layer and hence injective.

Now let $J_s = \sum_Y i_*H_S^*Y \subset H_S^*M$, where the sum is over all fixed-point components of all subtori $B \subset S$ of rank $\geq s$. Note that J_s is an ideal, by the ? property of the pushforward.

Proposition 5.5 $J_s = \mathcal{F}_s$.

Proof: We have already seen that $J_s \subset \mathcal{F}_s$. This is clear anyway from naturality of the pushforward, since each Y occuring in the proposition is disjoint from the open submanifold M_0^{s-1}. The reverse inclusion follows by downward induction on s.

It's interesting to see how two theorems on transfer of Carlson and Totaro, respectively, are reflected in the S-manifold setting. In its simplest form, Carlson's theorem on transfer says that for a finite p-group, the ideal generated by the images of transfers from all proper subgroups has the same radical as the kernel of restriction to the maximal central p-torus. To state an analogous theorem for S-manifolds, let E denote the kernel of the S-action on M, $rank\, E = e$. Since M is connected, there is a unique S-homotopy class of maps $\phi : S/E \longrightarrow M$. Let K denote the kernel of $\phi^* : H_S^*M \longrightarrow H_S^*(S/E) = H_E^*$.

Proposition 5.6 *Let $I \subset H_S^* M$ denote the ideal generated by all $i_* H_S^Y$, Y ranging over fixed-point components of p-subtori B such that $Y \neq M$ and $Y_B \neq \emptyset$. Then $rad\, I = rad\, K$.*

Proof: First of all, I is none other than J_{e+1}, which by the preceeding proposition is $\mathcal{F}_{e+1} = Ker\,(H_S^* M \longrightarrow H_S^* M_e^e$. Hence $I \subset K$ and $rad\, I \subset rad\, K$. For the reverse inclusion, it suffices to show that if $\alpha \in K$, then α restricted to M_e^e is nilpotent. But $H_S^* M_e^e \cong H_E^* \otimes N$ for a finite dimensional algebra N, and since $\alpha \in K$ it restricts to an element of the nilpotent ideal $H_E^* \otimes N_{>0}$.

In its simplest form, Totaro's theorem on transfer says that for a finite p-group G, the quotient of H_G^* by Carlson's ideal above is Cohen-Macaulay of dimension c (here c as usual is the rank of a maximal central p-torus). An analogue for S-manifolds reads as follows, keeping the notation of Proposition 5.6:

Proposition 5.7 *$H_S^* M/I$ is Cohen-Macaulay of dimension e.*

Proof: Since $I = J_{c+1}$, by Proposition 5.5 we have

$$H_S^* M/I = H_S^* M/\mathcal{F}_{e+1} = H_S^* M_e^e = H_E^* \otimes N.$$

Clearly $dim\,(H_E^* \otimes N) = e = depth\,(H_E^* \otimes N)$, and the result follows.

The theorems of Carlson and Totaro are actually stronger in two ways: (i) they are stated and proved for general finite groups; and (ii) they show that analogous statements hold using only transfers from proper centralizers of p-tori. It appears to be difficult to deduce the Carlson and Totaro theorems from the S-manifold versions using exchange-descent. Of course, what is needed first is an analysis of how exchange and descent interact with the transfer. This could be an interesting topic for further investigation.

We conclude this section by showing that in the case $M = G \backslash U$, the injectivity in Proposition 5.4 holds in a stronger form. The notation is as above.

Proposition 5.8 *Suppose $K := Y - Y_B$ has even codimension in Y. Then the normal bundle ν_Y of Y in M is orientable, and its Euler class $e(\nu_Y)$ is a non-zerodivisor in $H_S^* Y$.*

Proof: To show that ν_Y is orientable, consider the first Stiefel-Whitney class $w_1(\nu_Y) \in H^1(Y; \mathbb{F}_2)$. We have seen that ν_{Y_B} is orientable; hence $w_1(\nu_Y)$ restricts to zero on Y_B. From the localization sequence $H^1(Y|K; \mathbb{F}_2) \longrightarrow H^1(Y; \mathbb{F}_2) \longrightarrow H_1(Y_B; \mathbb{F}_2)$ we see that it is enough to show $H^1(Y|K; \mathbb{F}_2) = 0$. This should follow from a duality isomorphism $H^1(Y|K) \cong H^{n-1} K = 0$, assuming the pair (Y, K) satisfies the appropriate hypotheses. Alternatively, elements of $H_1(Y|K)$ are represented by sums of smooth relative 1-cycles of the form $\alpha : [0,1] \longrightarrow Y$ whose endpoints lie in Y_B. Since by assumption all fixed-point components of all p-tori properly containing B have codimension at least 2, by transversality we can find a sequence of smooth homotopies rel endpoints moving α away from each fixed-point component successively. Hence $H^1(Y|K; \mathbb{F}_2) \cong H_1(Y|K; \mathbb{F}_2) = 0$.

The set of zerodivisors in $H_S^* Y$ is the union of the associated primes, which all have the form $\mathfrak{p}_{D,N}$ for some p-torus D with $B \subset D \subset S$ and component N of Y^D. Since $\mathfrak{p}_{D,N} \subset \mathfrak{p}_{B,Y}$, if $e(\nu_Y)$ is a zerodivisor then it lies in $\mathfrak{p}_{B,Y}$, i.e. restricts to zero along any equivariant map $S/B \longrightarrow Y$. But this is false; we saw in ...

6 Associated primes

It is an open problem to determine the associated primes of H_G^*. The first result in this direction is the theorem of Duflot, which says that all associated primes are p-toral:

Theorem 6.1 *Let $\mathfrak{p}$ be an associated prime of H_G^*. Then $\mathfrak{p} = \mathfrak{p}_A$ for some p-torus $A \subset G$.*

To see this, we first prove the analogous statement for S-manifolds (subject to the finiteness condition stated in the introduction).

Theorem 6.2 *Let M be a smooth S-manifold, $\mathfrak{p}$ an associated prime of H_S^*M. Then $\mathfrak{p} = \mathfrak{p}_{B,Y}$ for some $B \subset S$ and Y a component of M^B.*

Proof: Recall that if R is a commutative ring and $0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$ is a short exact sequence of R-modules, then $Ass_R M \subset Ass_R M_1 \cup Ass_R M_2$. Hence in the notation of Theorem 5.3, $\mathfrak{p}$ is an associated prime of the H_S^*M-module H_S^*W for some W, where $W = Y_B$ for some B, Y. Since H_S^*W is the tensor product of the polynomial part of H_S^*B and a finite-dimensional algebra, as a module over itself its only associated prime is $\mathfrak{p}_{B,W}$. It follows that $\mathfrak{p} = \mathfrak{p}_{B,Y}$.

Proof of Theorem6.1: Under exchange, p-toral primes correspond to p-toral primes; i.e., the primes $\mathfrak{p}_{B,Y}$ of $H_S^*G\backslash U$ correspond to the primes $\mathfrak{p}_{X,A}$ of H_G^*U/S. Hence by Theorem 6.1, all associated primes of H_G^*U/S are p-toral. Since $H_G^* \longrightarrow H_G^*U/S$ is faithfully flat, the induced map on *Spec* preserves associated primes and in fact induces a surjection $Ass\, H_G^*U/S \longrightarrow Ass\, H_G^*$.(ref). Hence if $\mathfrak{p} \in Ass\, H_G^*$, there is a p-toral prime $\mathfrak{p}_{A,X}$ lying over it in H_G^*U/S. It follows that $\mathfrak{p} = \mathfrak{p}_A$, as desired.

There is a further restriction on which p-tori can occur.

Theorem 6.3 *If $\mathfrak{p}_A$ is an associated prime of H_G^*, then A is the maximal central p-torus of its centralizer C_GA. In particular, A contains the maximal central p-torus of G.*

The S-manifold analogue of this result follows immediately from the Duflot filtration. The point is that associated primes in H_S^*M are associated primes of direct summands of the Duflot layers, and so have the form $\mathfrak{p}_{B,Y}$ *with Y_B nonempty.* Moreover, $Y_B \neq \emptyset \Leftrightarrow B$ is the kernel of the S-action on Y, hence:

Theorem 6.4 *If $\mathfrak{p}_{B,Y}$ is an associated prime of H_S^*M, then B is the kernel of the S-action on Y.*

We could at this point deduce Theorem 6.3 from Theorem 6.4 by descent and exchange, but we will prove a stronger result. If A is a p-torus in G, then it is also one in C_GA. The corresponding prime ideals will be denoted $\mathfrak{p}_A \subset H_G^*$ and $\mathfrak{p}'_A \subset H_{C_GA}^*$.

Theorem 6.5 *The following are equivalent:*

a) $\mathfrak{p}_A \in Ass\, H_G^*$;
b) $\mathfrak{p}'_A \in Ass\, H_{C_GA}^*$;
c) $depth\, H_{C_GA}^* = rank\, A$.

Assuming Theorem 6.5, we prove Theorem 6.3: Suppose $\mathfrak{p}_A$ is an associated prime of H^*_G. If A is the not the maximal central p-torus of $C_G A$, then by Duflot's theorem $depth\, H^*_{C_G A} > rank\, A$, contradicting Theorem 6.5.

As usual, we first prove the S-manifold analogue and then deduce Theorem 6.5 by exchange and descent.

Theorem 6.6 *The following are equivalent:*

a) $\mathfrak{p}_{B,Y} \in Ass\, H^*_S M$*;*
b) $\mathfrak{p}'_{B,Y} \in Ass\, H^*_S Y$*;*
c) $depth\, H^*_S Y = rank\, B$*.*

Proof: (a) $\Leftrightarrow$ (b): Consider the short exact sequence of $H^*_S M$-modules

$$0 \longrightarrow \Sigma^{cd(Y)} H^*_S Y \longrightarrow H^*_S M \longrightarrow N \longrightarrow 0,$$

where the first map is the pushforward and N is its cokernel. Since distinct pairs (B, Y) define distinct prime ideals [ref], $\mathfrak{p}_{B,Y}$ is not an associated prime of N (see Prop 6.4...). Hence $\mathfrak{p}_{B,Y} \in Ass\, H^*_S M \Leftrightarrow \mathfrak{p}_{B,Y} \in Ass_{H^*_S M} H^*_S Y$. Next, by commutative algebra there is a surjection

$$Ass_{H^*_S Y} \longrightarrow Ass_{H^*_S M} H^*_S Y.$$

Putting these facts together, the equivalence of (a) and (b) follows easily.

(b) $\Leftrightarrow$ (c): The depth of a ring is always $\leq$ the minimal dimension of an associated prime. So if (b) holds, then $depth\, H^*_S Y \leq rank\, B$. Since B acts trivially on Y, $depth\, H^*_S Y \geq rank\, B$. Thus (b) $\Rightarrow$ (c).

Conversely, suppose $depth\, H^*_S Y = rank\, B$, and write $S = B \times B'$. Then by the Kunneth formula for local cohomology we have $depth\, H^*_{B'} Y = 0$; in other words, the maximal graded ideal $\mathfrak{m}$ is an associated prime of $H^*_{B'} Y$. Thus $\mathfrak{m} = ann\, \alpha$ for some $\alpha \in H^*_{B'} Y$. If $p = 2$, we conclude that $\mathfrak{p}_{B,Y} = ann\, (1 \otimes \alpha)$ and so is associated; a similar argument works at odd primes.

Proof of Theorem 6.5: We show (b) $\Leftrightarrow$ (c) and leave the rest to the reader. Suppose $\mathfrak{p}'_A$ is associated. Then by commutative algebra, $rank\, A = dim\, \mathfrak{p}'_A \geq depth\, H^*_{C_G A}$. By Duflot's theorem $depth\, H^*_{C_G A} \geq rank\, A$, so we have equality.

Conversely, suppose $depth\, H^*_{C_G A} = rank\, A$. Choosing the maximal torus so that $A \subset S$ (alternatively, we can forget about G at this point and just start from a suitable representation of $C_G A$), let X be any component of $(U/S)^A$. Then $depth\, H^*_{C_G A} X = depth\, H^*_{C_G A} = rank\, A$. Under exchange we then have $depth\, H^*_S Y = rank\, B$ and hence $\mathfrak{p}'_B$ is associated by Theorem 6.6. Then by exchange and descent $\mathfrak{p}'_A$ is associated, as desired.

7 Carlson's theorem and Carlson's depth conjecture

Theorem 7.1 *Let $depth\, H^*_G = s$. Then H^*_G is detected on $\{C_G A : rank\, A = s\}$.*

Note that the theorem has no content when $s = c$ is the Duflot minimum, since then G itself is one of the detecting subgroups. Note also the equivalent form: If $depth\, H^*_G \geq s$, then H^*_G is detected on $\{C_G A : rank\, A \geq s\}$.

The S-manifold version of this theorem is:

Theorem 7.2 *Let $depth\, H^*_S M = s$. Then $H^*_S M$ is detected on $\{H^*_S Y : rank\, Y = s\}$ (i.e. on fixed-point components of rank s subgroups).*

Proof: Suppose there is a nonzero element $a \in H^*_S M$ such that for all B of rank s and all components Y of M^B, $i^*_Y a = 0$. By the ? property of the pushforward, $(i_* H^*_S Y)a = 0$. Hence $J_s a = 0$ (recall ...). Then there is an associate prime $\mathfrak{p}$ with $J_s \subset \mathfrak{p}$. By ? $dim\, \mathfrak{p} \leq dim J_s < s$. Hence $depth\, H^*_S M < s$.

Proof of Theorem 7.1: There is a commutative diagram

$$\begin{tikzcd} H^*_G \arrow[r] \arrow[d] & \oplus H^*_{C_G A} \arrow[d] \\ H^*_G(U/S) \arrow[r] & \oplus H^*_{C_G A} X \end{tikzcd}$$

where A ranges over rank s p-tori and X ranges over fixed-point components of such tori in U/S. The bottom arrow is injective by exchange and the previous theorem, while the left arrow is always injective. Hence the top arrow is injective, as desired.

Carlson conjectured that the converse of Theorem 7.1 is true: If H^*_G is detected on $\{C_G A : rank\, A \geq s\}$, then $depth\, H^*_G \geq s$. He also made the related conjecture that there is always an associated prime $\mathfrak{p}$ with $dim\, \mathfrak{p} = depth\, H^*_G$. As Carlson notes, this does not hold for general rings.

The following special case of Carlson's conjecture was proved by Green for p-groups, and by Kuhn in the general case. Recall that c is the rank of the maximal central p-torus C.

Theorem 7.3 *If H^*_G is detected on $\{C_G A : rank\, A \geq c + 1\}$, then $depth\, H^*_G \geq c + 1$.*

Proof: Suppose $depth\, H^*_G < c+1$. Then by Duflot's theorem, $depth\, H^*_G = c$. By Theorem 6.5, it follows that $\mathfrak{p}_C$ is an associated prime, say $\mathfrak{p}_C = ann\, x$. We will show that x restricts to zero on $C_G A$ with $rank\, A \geq c + 1$.

Let A have rank $c+1$ and let $i : C_G A \longrightarrow G$ denote the inclusion. Suppose $i^* x \neq 0$. Since $\mathfrak{p}_C i^* x = 0$, there exists $\mathfrak{q} \in Ass_{H^*_G} H^*_{C_G A}$ with $\mathfrak{p}_C \subset \mathfrak{q}$. By commutative algebra there exists $\mathfrak{q}' \in Ass H^*_{C_G A}$ mapping to $\mathfrak{q}$ under the induced map $Spec\, H^*_{C_G A} \longrightarrow H^*_G$. We then have

$$dim\, \mathfrak{q}' = dim\, \mathfrak{q} \leq dim\, \mathfrak{p}_C = c,$$

where the first equality holds since i^* is a finite morphism. But $depth\, H^*_{C_GA} > c$ by Duflot's theorem, so this is a contradiction (an associated prime cannot have dimension less than the depth).

8 Castelnuovo-Mumford regularity

If A is a 2-torus of rank r, then $H^*_A \cong \mathbb{F}_2[x_1, ..., x_r]$ with $x_i = 1$ for all i. Hence $reg\, A = 0$ as shown in the appendix. If A is a p-torus with p-odd, then $H^*_A \cong \mathbb{F}_p[y_1, ..., y_i] \otimes \mathbb{F}_p\langle x_1, ..., x_r\rangle$ with $|y_i| = 2$ and $|x_i| = 1$ for all i. Then the polynomial part has regularity $-n$ and the exterior part has regularity n, so by the tensor product formula $reg\, H^*_A = -n + n = 0$. Again, see the appendix for background.

The following striking result was conjectured by Dave Benson:

Theorem 8.1 *If G is any finite group, $reg\, H^*_G = 0$.*

Benson himself proved that $reg\, H^*_G \geq 0$, using a spectral sequence of Greenlees. Symonds proved the reverse inequality by descent and exchange, as follows:

Let $n = dim\, U = dim\, U/S = dim\, G\backslash U$. Then

$$reg\, H^*_G + n = reg\, H^*_G(U/S) = reg\, H^*_S(G\backslash U).$$

To see the first equality, first note that by the Independence Theorem we can compute the local cohomology of $H^*_G(U/S)$ by regarding it as an H^*_G-module. This yields the first equality because $H^*_G(U/S)$ is free over H^*_G with top generator in degree n. The second equality is immediate by the exchange isomorphism. Thus it suffices to show that for a smooth S-manifold M of dimension n, we have $reg\, H^*_S M \leq n$.

Again by the Independence Theorem, we can compute local cohomology of $H^*_S M$ by regarding it as an H^*_S-module. From the Duflot filtration, taking into account the dimension shifts occuring in the Duflot layers, we have at once that

$$reg\, H^*_S M \leq reg\, H^*_S Y_B + cd(Y)$$

for all subtori B and all components Y of M^B with $Y_B \neq \emptyset$. Moreover, choosing B' so that $S = B \times B'$ and hence $H^*_S = H^*_B \otimes H^*_{B'}$, we know that B' acts freely on Y_B and

$$H^*_S Y_B \cong H^*_B \otimes H^*(Y_B/B').$$

Since Y_B/B' is a smooth manifold of dimension $d(Y)$, its cohomology vanishes above dimension $d(Y)$. Hence

$$reg\, H^*_S Y = reg\, H^*_B + reg\, H^*(Y_B/B') \leq d(Y)$$

and therefore $reg\, H^*_S M \leq d(Y) + cd(Y) = n$ as desired.

Remark. The Duflot summand $\Sigma^{cd(Y)} H^*_S Y_B$ achieves the maximal possible regularity n precisely when Y_B/B' has non-vanishing cohomology in dimension $d(Y)$. This is possible only when Y_B is compact, and hence $Y = Y_B$. Thus (Y, B) is maximal in the poset of fixed-point components. In a general S-manifold this need not imply that B is maximal in the poset of all isotropy groups. But in the case $M = G\backslash U$, B is maximal and hence the corresponding p-torus A is a maximal p-torus of G.

Symonds' theorem on regularity yields upper bounds on the dimensions of minimal generating sets for H_G^* as an algebra, and on the relations between these generators. Previously, no upper bounds were known. In particular, he shows that H_G^* is generated as an algebra by elements of grade $\leq |G| - 1$, and the relations between these generators are generated by elements of grade $\leq 2(|G| - 1)$. See [Symonds].

One last remark: Symonds gives an example due to Jean Lannes of a finite 1-dimensional $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-complex X whose equivariant cohomology has regularity three. Thus the regularity theorem for S-manifolds does not extend to general complexes.

9 Local cohomology and the Duflot complex

By the Independence Theorem (Appendix A), we can compute the local cohomology of H_S^*M by regarding it as an H_S^*-module, or even as a P_S-module, where P_S is the polynomial part of H_S^*. Applying local cohomology to the tower

(display)

we get a spectral sequence going from the local cohomology of the layers to the local cohomology of H_S^*M. However, the local cohomology of the i-th layer is concentrated in cohomological degree i, so this spectral sequence degenerates to a cochain complex:

$$\mathfrak{h}^0 H_S^* M_0^0 \longrightarrow \mathfrak{h}^1 H_S^* M_1^1 \longrightarrow \ldots \longrightarrow \mathfrak{h}^r H_S^* M_r^r.$$

The i-th cochain module is a direct sum of modules of the form $\mathfrak{h}^i H_S^* Y_B$, where $rank\, B = i$. The latter module in turn has the form

$$\mathfrak{h}^i(P_B) \otimes N = \Sigma^{-d} P_B^* \otimes N$$

for a certain finite dimensional module N, with $d = 2i$ for p odd and $d = i$ for $p = 2$.

Taking linear duals we get a corresponding Duflot chain complex whose homology is the local homology of H_S^*M. Now the summands of the i-th chain group have the form $\Sigma^d P_B \otimes N^*$. Thus we have a finitely-generated free P_B-module, regarded as a P_S-module by via the restriction homomorphism $P_S \longrightarrow P_B$. In particular, the i-th chain group has Krull dimension i as a P_S-module.

It is interesting to note that certain general theorems of commutative algebra can be proved directly for H_G^* using the Duflot complex. For example, Grothendieck's Theorem asserts that the Krull dimension $dim\, Q$ of a finitely-generated module Q is the largest i for which $\mathfrak{h}^i Q \neq 0$. Here the vanishing for $i > dim\, H_S^*M = r$ is immediate, since the Duflot complex is zero above dimension r. Moreover $\mathfrak{h}_r H_S^*M$ is nonzero and in fact of Krull dimension r, because it is the kernel of a homomorphism from an r-dimensional module to an $(r-1)$-dimensional module. A trivial application of exchange and descent yields the result for H_G^*.

Another example is a theorem of [Sharp], asserting (under suitable hypotheses that apply here) that if a module has an associated prime $\mathfrak{p}$ of dimension i, then its i-th local cohomology is nonzero [and in fact the i-th homology has $\mathfrak{p}$ as an associated prime]. Before either of us was aware of Sharp's theorem, the first author proved this result for H_G^* as follows:

By the usual exchange and descent argument, we reduce to the analogous result for H_S^*M. Suppose $\mathfrak{p}_{B,Y}$ is an associated prime of dimension i. Then in the notation of Theorem 6.6, $\mathfrak{p}'_{B,Y}$ is associated in H_S^*Y, and $depth\, H_S^*Y = rank\, B$. Hence $\mathfrak{h}^i H_S^*Y$ is nonzero, and by the Kunneth theorem for local cohomology we have

$$\mathfrak{h}^i H_S^*Y \cong \mathfrak{h}^i H_B^* \otimes \mathfrak{h}^0 H_{S/B}^* Y$$

with an analogous isomorphism in local homology. In particular, the H_S^*-module $\mathfrak{h}_i H_S^*Y$ has Krull dimension i. By Theorem 5.4, there is a short exact sequence

$$0 \longrightarrow \Sigma^{cd(Y)} H_S^*Y \longrightarrow H_S^*M \longrightarrow N \longrightarrow 0,$$

compatible with the Duflot filtration and the direct sum decomposition of the layers. Applying local homology yields an exact sequence

$$\mathfrak{h}_i H_S^*M \longrightarrow \mathfrak{h}_i \Sigma^{cd(Y)} H_S^*Y \xrightarrow{\partial} \mathfrak{h}_{i-1} N.$$

As shown earlier (ref), $\mathfrak{h}_{i-1}N$ has Krull dimension at most $i-1$. Hence $Ker\,\partial$ has dimension i and therefore $\mathfrak{h}_i H_S^*M$ has dimension i.

10 Appendix A: Local cohomology

For more information on local cohomology, see [Iyengar et. al.] or [Brodmann-Sharp]. Here we will just give the definition and a few basic properties that we need. For an introduction to Castelnuovo-Mumford regularity, with some history, see [Eisenbud].

Let R be a commutative ring, I an ideal in R. If M is an R-module, the I-torsion $\Gamma_I(M)$ is defined by

$$\Gamma_I(M) = \{x \in M : I^n x = 0 \text{ for some } n\}.$$

Γ_I is a left exact functor, whose right derived functors $\mathfrak{h}_I^*M$ are called *local cohomology with respect to I*.

We will work in the graded setting. In fact for us, R is always a connected graded noetherian algebra over a field κ, and I is always the maximal graded ideal $\mathfrak{m}$ of positive dimensional elements. So we will drop the subscript and write $\mathfrak{h}^*M$ for $\mathfrak{h}_\mathfrak{m}M$. Note also that each functor $\mathfrak{h}^i$ is a graded R-module. We write $\mathfrak{h}^{i,j}$ for the j-th grade of $\mathfrak{h}^i$, and call this the *internal grading* of $\mathfrak{h}^i$. If necessary for clarity, we write $\mathfrak{h}_R^*M$ to indicate that we are regarding M as an R-module.

A simple but very helpful example is to take $R = M = \kappa[x]$, where $|x| = d$. As graded modules there is a length one injective resolution

$$\kappa[x] \longrightarrow \kappa[x, x^{-1}] \longrightarrow \kappa[x]/x^{\infty},$$

where $\kappa[x]/x^{\infty}$ is just the cokernel of the first map. Hence $\mathfrak{h}^1\kappa[x] = \kappa[x]/x^{\infty}$, and all other local cohomology groups are zero. Note that $\mathfrak{h}^1$ is negatively graded, with its highest nonzero grade $\mathfrak{h}^{1,-d}$. It is an $\mathfrak{m}$-torsion module and is not finitely-generated. Many of the properties discussed below are already illustrated in this simple example.

Proposition 10.1 *a) $\mathfrak{h}^i M$ is an $\mathfrak{m}$-torsion module.*
b) If M is $\mathfrak{m}$-torsion, then $\mathfrak{h}^i M = 0$ for $i > 0$.

Here (a) is obvious, while for (b) one shows that a torsion module admits an injective resolution by torsion modules. Note that any M that is bounded above is $\mathfrak{m}$-torsion.

Now suppose M_1, M_2 are modules over R_1, R_2. Then we can form the $(R_1 \otimes R_2)$-module $M_1 \otimes M_2$, and there is a Kunneth theorem:

Proposition 10.2

$$\mathfrak{h}^{m,n}_{R_1 \otimes R_2}(M_1 \otimes M_2) \cong \oplus_{i+j=m, r+s=n} \mathfrak{h}^{i,r}_{R_1} M_1 \otimes \mathfrak{h}^{j,s}_{R_2} M_2$$

Example. Consider a polynomial algebra $R = \kappa[x_1, ..., x_n] = \otimes_{i=1}^{n} \kappa[x_i]$, where $|x_i| = d_i$. Then $\mathfrak{h}^* R = 0$ for $i \neq n$ and

$$\mathfrak{h}^n R \cong \kappa[x_1]/x_1^{\infty} \otimes ... \otimes \kappa[x_n]/x_n^{\infty}.$$

In particular the highest nonzero grade is $\mathfrak{h}^{n,-d}$, where $d = \sum d_i$.

The next result is known as the “Independence Theorem”, since it shows that local cohomology in a certain sense independent of the base ring used to compute it. In fact the version we give here is only one corollary of the Independence Theorem, but it is the version we will use.

Proposition 10.3 *Let $\phi : R \longrightarrow R'$ be a finite ring homomorphism (of connected graded noetherian algebras), and let M be an R'-module. Then there is a natural isomorphism*

$$\mathfrak{h}^*_R M \cong \mathfrak{h}^*_{R'} M.$$

For example, we can compute $\mathfrak{h}^*_{R'} M$ by taking a polynomial subalgebra $R \subset R'$ over which R' is finite (Noether normalization) and doing the computation over R. In topological applications we often have a fibration $E \longrightarrow X$ with H^*E finite over H^*X; then we can compute the local cohomology of H^*E as a module over itself by computing it as a module over H^*X.

Theorem 10.4 *Suppose M is finitely-generated.*
a) $i = depth\, M$ is minimal such that $\mathfrak{h}^i \neq 0$.
b) $i = dim\, M$ is maximal such that $\mathfrak{h}^i \neq 0$.

Part (b) is Grothendieck's vanishing theorem. Note the theorem is not just saying that $\mathfrak{h}^i$ vanishes below the depth and above the dimension; it says also that $\mathfrak{h}^i$ is always nonzero for $i = depth\, M$, $i = dim\, M$.

The next result is a version of Grothendieck's duality theorem. We don't use duality in this paper, but according to [Iyengar] one of the main motivations for local cohomology was the possibility of proving such duality theorems. Let N^* denote the graded linear dual $Hom\,(N, \kappa)$ of a graded R-module N. In the applications N is finitely-generated and hence finite dimensional in each grade.

Theorem 10.5 *Let R be a polynomial ring $\kappa[x_1, ..., x_n]$, M a finitely-generated R-module. Then $\mathfrak{h}^i M \cong (Ext_R^{n-i}(M, \Sigma^d R))^*$.*

One can also define local homology $\mathfrak{h}_* M$ as the linear dual of cohomology: $\mathfrak{h}_i M = (\mathfrak{h}_i M)^* = Hom\,(\mathfrak{h}^{-i} M, \kappa)$. For example, if R is a polynomial ring on n variables then $\mathfrak{h}_n R \cong \Sigma^{d(R)} R^*$.

10.1 Castelnuovo-Mumford regularity

It is easy to show that if M is a finitely-generated R-module, then for each i, $\mathfrak{h}^i M$ is bounded above. Let $a_i(M)$ denote the maximal grade in which M is nonzero. Since only finitely many of the groups $\mathfrak{h}^i M$ are nonzero, we can define an integer $reg\, M$, the Castelnuovo-Mumford regularity of M, by

$$reg\, M = max\,\{a_i(M) + i\}.$$

Although rather obscure at first encounter, $reg\, M$ turns out to be useful e.g. for studying the dimensions of generators in minimal free resolutions. See Symonds' Theorem? above for an application to algebra generators of group cohomology rings.

Example. If M is bounded above (with our hypotheses, this is equivalent to finite-dimensional), then $\mathfrak{h}^0 M = M$ and the higher $\mathfrak{h}^i$'s are zero. Hence $reg\, M = 0$.

Example. If $R = \kappa[x_1, ..., x_n]$, with $|x_i| = d_i$, then as a module over itself, $\mathfrak{h}^i \neq 0$ only for $i = n$, as shown above. The calculation also shows that $reg\, R = -d + n$, where $d = \sum d_i$.

The following properties of regularity follow immediately from corresponding properties of local cohomology:

1. Regularity is "independent of the base ring" in the sense of Proposition 10.3.
2. In the situation of Proposition 10.2 (the Kunneth theorem),
   $reg\,(M_1 \otimes M_2) = reg\, M_1 + reg\, M_2$.
3. If $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is exact, then $reg\, M_2 \leq max\,(reg\, M_1, reg\, M_3)$.

11 Appendix B: Localization at a submanifold and the pushforward in equivariant cohomology

Let Z be a smooth S-manifold, $i = i_{Y,Z} : Y \subset Z$ a closed S-submanifold of codimension $d = d(Y, Z)$. Borrowing a notation from Hatcher, we write $H^*_S(Z|Y) := H^*_S(Z, Z - Y)$ for cohomology localized at Y. Note $H^*_S Z = H^*_S(Z|Z)$. It has various variances:

1. It is covariant with respect to inclusions in Y: $X \subset Y$ induces $H^*_S(Z|X) \longrightarrow H^*_S(Z|Y)$. This is a map of $H^*_S Z$-modules.

2. It is contravariant with respect to inclusions in Z: $Z' \subset Z$ induces $H^*_S(Z|Y) \longrightarrow H^*_S(Z'|Y)$, again a map of $H^*_S Z$-modules.

3. More generally, it is contravariant with respect to pair inclusions $(Z', Y') \subset (Y, Z)$ provided that $Y' = Y \cap Z'$. Since we are limiting our attention to closed submanifolds, we need to also assume the intersection is transverse if we want to stay in that context.

If Y_1, Y_2 are closed S-submanifolds that intersect transversally (again the transversality is just to stay in the manifold context), there is a cup product

$$H^*_S(Z|Y_1) \otimes H^*_S(Z|Y_2) \longrightarrow H^*_S(Z|(Y_1 \cap Y_2)).$$

Next we have the usual Gysin/push-forward/transfer map $i_* : \Sigma^d H^*_S Y \longrightarrow H^*_S Z$, defined as the composite

$$\Sigma^d H^*_S Y \xrightarrow{\cong} H^*_S(Z|Y) \xrightarrow{p^*} H^*_S Z,$$

where the first map is Thom isomorphism plus excision and the second the natural map for the pair. Among the properties satisfied by i_*, we mention the following: First, note that i_*1 is the class obtained by pulling back the Thom class of the normal bundle in the usual way.

Proposition 11.1 *a) $i^* i_* 1 = e(\nu_Y)$, where e is the Euler class and ν_Y is the normal bundle.*
b) $i_*((i^*a)b) = a i_* b$ (this is just the module map property restated).

Proposition 11.2 *Naturality.*
Suppose U is an S-invariant subset of Zthat is either open or a closed submanifold transverse to Y, and consider the commutative diagram of inclusions:

$$\begin{tikzcd} U \cap Y \arrow[r, "j'"] \arrow[d, "i'"'] & Y \arrow[d, "i"] \\ U \arrow[r, "j"'] & Z \end{tikzcd}$$

Then $j^* i_* = i'_* j'^*$.

Proof: This follows from naturality of the Thom isomorphism and of the maps p associated to the pairs.

Note the trivial special case: If $U \cap Y = \emptyset$, $j^* i_* = 0$.

Proposition 11.3 *Functoriality*

Given inclusions $X \xrightarrow{j} Y \xrightarrow{i} Z$ *of closed* S*-submanifolds,* $(i \circ j)_* = i_* \circ j_*$.