UPPER SEMICONTINUITY OF KLV POLYNOMIALS FOR CERTAIN BLOCKS OF HARISH-CHANDRA MODULES

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To David Vogan on his 60th birthday

ABSTRACT. We show that the coefficients of Kazhdan-Lusztig-Vogan polynomials attached to certain blocks of Harish-Chandra modules satisfy a monotonicity property relative to the closure order on K-orbits in the flag variety.

Let G be a complex connected reductive group with Lie algebra $\mathfrak{g}$ and Borel subgroup B. Recall that the flag variety G/B decomposes into finitely many B-orbits $\mathcal{O}_w$, which are indexed by elements w of the Weyl group W of G. The closure $\bar{\mathcal{O}}_w$ of an orbit $\mathcal{O}_w$ is called a Schubert variety. Kazhdan and Lusztig introduced polynomials $P_{v,w}$ in one variable q, indexed by pairs v, w of elements in W in [KL79], which they later showed measured the singularities of Schubert varieties [KL80]. More precisely, they showed that the coefficient c_i of q^i in P_{w_0w,w_0v} satisfies

$$c_i = \dim IH^{2i}_x(\bar{\mathcal{O}}_w; \bar{\mathbb{Q}}_p)$$

for any $x \in \mathcal{O}_v$, where the right side denotes the local $2i$th intersection cohomology group with values in the constant sheaf $\mathbb{Q}_p$, p is a prime and w_0 is the longest element of W [KL80]. A fundamental result of Irving, first proved in [I88], using results of [GJ81], asserts that the singularities of $\bar{\mathcal{O}}_w$ increase as one goes down; more precisely, if $c_i^{v,w}$ denotes the coefficient of q_i in $P_{v,w}$, then $c_i^{v,w} \geq c_i^{v',w}$ whenever $v \leq v' \leq w$ in the Bruhat order on W. (Following Li and Yong [LY10], we call this property upper semicontinuity.) Irving's proof uses representation theory; later Braden and MacPherson gave a geometric argument in [BM01]. The purpose of this note is to establish the corresponding inequality in some cases for coefficients of Kazhdan-Lusztig-Vogan (KLV) polynomials and closures of K-orbits in G/B, where K is a symmetric subgroup of G. So let θ be an involutive automorphism of G and K the fixed points of θ acting on G. If $\mathcal{O}, \mathcal{O}'$ are two K-orbits in G/B with $\bar{\mathcal{O}} \subset \bar{\mathcal{O}}'$ and if γ, γ' are K-equivariant sheaves of one-dimensional $\bar{\mathbb{Q}}_p$ vector spaces on $\mathcal{O}, \mathcal{O}'$, respectively, then Lusztig and Vogan have constructed a polynomial $P_{\gamma,\gamma'}$ such that the coefficient d_i of q^i in $P_{\gamma,\gamma'}$ equals the dimension of the local $2i$th intersection cohomology sheaf of the Deligne-Goresky-MacPherson extension of γ' to the closure of $\mathcal{O}'$, supported at a point in $\mathcal{O}$ [LV83, V83]. If γ and γ' are trivial then we write $P_{\mathcal{O},\mathcal{O}'}$ instead of $P_{\gamma,\gamma'}$. We then ask under what conditions, given three orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ with $\bar{\mathcal{O}}_1 \subset \bar{\mathcal{O}}_2 \subset \bar{\mathcal{O}}_3$, do we have

$$d_i^{\mathcal{O}_1,\mathcal{O}_3} \geq d_i^{\mathcal{O}_2,\mathcal{O}_3}$$

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for all i, where the terms denote the coefficients of q^i in the polynomials corresponding to the pair of orbits in the superscripts. In general, this fails, for example if $G = SL(4)$. The problem stems from the existence of nontrivial sheaves γ, even though the inequality is stated for trivial γ only. Now we can state our result.

Theorem. *With notation as above, assume that all K-orbits $\mathcal{O}$ admit only the trivial sheaf (equivalently, all orbits $\mathcal{O}$ are simply connected, or all Cartan subgroups of the real form G_0 of G corresponding to K are connected). If $\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3$ then $d_i^{\mathcal{O}_1,\mathcal{O}_3} \geq d_i^{\mathcal{O}_2,\mathcal{O}_3}$.*

Proof. The proof follows Irving's proof in [I88] for Schubert varieties closely. We begin by observing that the hypothesis on G and K implies that all roots of $\mathfrak{g}$ relative to any K-orbit (or to the corresponding θ-stable Cartan subgroup of and choice of positive roots for G_0) are complex or noncompact imaginary type I, whence the recursion formulas of [V83, 6.14] imply that the constant term of $P_{\mathcal{O},\mathcal{O}'}$ equals 1 whenever $\bar{\mathcal{O}} \subseteq \bar{\mathcal{O}}'$.

Now we appeal to representation theory. There is a single block B containing all simple $(\mathfrak{g}, K)$-modules of trivial infinitesimal character. Vogan has shown in [V82] that there is another (possibly disconnected) complex group G' with symmetric subgroup K' and Lie algebra $\mathfrak{g}'$ and a block B' of $(\mathfrak{g}', K')$-modules, which we may take to have trivial infinitesimal character, such that there is a bijection between the set of K-orbits in G/B and a set S' of sheaves γ over K'-orbits in G'/B' that is order-reversing on the level of orbits. (The sheaves in S' parametrize the irreducible modules in B'.) In particular, since there is a unique maximal (open) K-orbit in G/B, there is a unique orbit minimal among the orbits corresponding to the sheaves in S'. Moreover, the values of the KLV polynomials for B at 1 count the multiplicities of composition factors in standard $(\mathfrak{g}', K')$-modules for B'; both the irreducible and the standard modules in B' are indexed by elements of S'. Casian and Collingwood have refined this result along the lines of the Gabber-Joseph refinement of the Kazhdan-Lusztig conjecture for Verma modules: they showed that the coefficients of KLV polynomials count multiplicities of composition factors in the socle filtrations of standard modules in B' [CC89], with standard modules indexed by sheaves on lower orbits occurring further down than those indexed by sheaves on higher orbits. (Here the standard modules in B' are normalized to have unique irreducible quotients, not unique irreducible submodules. Actually, Casian and Collingwood state their result for the weight filtration, but the proof of the main result of [I88] shows that this coincides with the socle filtration.) In particular, since all KLV polynomials for B have constant term 1, all standard modules in B' have isomorphic simple socles: in each case the socle is the unique simple standard module in B'.

We now show inductively that whenever γ, γ' are two elements of S', corresponding to orbits $\mathcal{O}, \mathcal{O}'$ in G'/B' with $\mathcal{O} \subset \mathcal{O}'$, then the standard module X_γ indexed by γ embeds in $X_{\gamma'}$ (we emphasize again that X_γ has a simple quotient). We have just shown this if X_γ is simple. In general we repeatedly apply the circle action of W defined in [V83, §5], moving up from the lowest sheaf in the partial order induced by the closure order on K'-orbits for B'. All simple roots α in question are complex or type I noncompact imaginary, so in every case the set $s \circ \gamma$ is a singleton for every sheaf γ in S', where s is the corresponding simple reflection to α. By Propositions 8.2.7 and 8.4.5 of [V81], whenever we apply the wall-crossing functor $T_\alpha := \phi_\alpha\psi_\alpha$ to X_γ we get a module having X_γ as a submodule with quotient $X_{s\circ\gamma}$. Hence whenever we have an inclusion $X_\gamma \subset X_{\gamma'}$ for $\gamma, \gamma' \in S'$, the induced inclusion $T_\alpha(X_\gamma) \subset T_\alpha(X'_\gamma)$ sends the copy of X_γ in $T_\alpha(X_\gamma)$ to the copy of $X_{\gamma'}$ in $T_\alpha(X_{\gamma'})$. Passing to the quotient of $T_\alpha(X_\gamma)$ by the image of X_γ inside it, we get a map from $X_{s\circ\gamma}$ to $X_{s\circ\gamma'}$, which is injective since there is exactly one copy of the unique irreducible standard module in B' inside any standard module X in B', at the lowest level of the socle filtration, and exactly two copies of the irreducible standard module in $T_\alpha(X)$, one at the lowest and the other at the next-to-lowest level of the socle filtration (so that the induced map from $X_{s\circ\gamma}$ to $X_{s\circ\gamma'}$ is injective on the socle of the former module and so injective). Now Definition 5.8 and Lemma 5.9 of [V83] ensure that every order relation among the elements of S' arises from one involving the lowest element by repeated applications of the circle action.

We now conclude the proof in exactly the same way that Irving did in the Schubert variety case [I88, Corollary 4]: $d_i^{\mathcal{O}_1,\mathcal{O}_3}$ counts the multiplicity of a suitable composition factor in the $2i$th level of the socle filtration of a suitable standard module X for $(\mathfrak{g}', K')$, while $d_i^{\mathcal{O}_2,\mathcal{O}_3}$ counts the multiplicity of the same composition factor in the $2i$th level of the socle filtration of a submodule of X. The desired inequality follows at once from the definition of the socle filtration. $\square$

We remark that the theorem extends to KLV polynomials for principal blocks (containing the trivial representation) of $(\mathfrak{g}, K)$-modules even if not all Cartan subgroups of the real form G_0 of G are connected, provided that there is only a single conjugacy class of disconnected Cartan subgroups and the groups in this class have only two components. This covers the cases $G = GL(2p), K = GL(p) \times GL(p); G = SO(2n), K = GL(n); G = E_7, K = E_6 \times \mathbb{C}$. On the other hand, the case $G = G_2, K = SL(2) \times SL(2)$ does not quite work: there is only one conjugacy class of disconnected Cartan subgroups, but the groups in it have four components. Here the theorem holds for the nonprincipal block, which has only one simple module, but fails for the principal one.

We further remark that Collingwood and Irving have explored the properties of Harish-Chandra modules in the block B (rather than its dual block B'), in the special case where the real form G_0 has only one conjugacy class of Cartan subgroups. Here the standard modules do not satisfy inclusion relations corresponding to inclusion of orbit closures, but many other familiar properties of modules in category $\mathcal{O}$ do carry over [CI92].

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