CRAZY GAMES AND CRAZIER NUMBERS

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Most people learn to play lots of games while growing up, most of which are frankly not mathematical, but some of them are. For example, my sons learned a game called Target in elementary school. Starting from 0, the two players take turns adding 1, 2, or 3 at each step; the winner is the first player to reach 28, the target number. In a more sophisticated game called Nim, the two players start with a number of piles of counters; each players in turn removes as many counters as he wants (but at least one) from one pile. The winner is the one who removes the last counter. There are also mathematical games with words, such as Ghost: players take turns adding a letter to the right of a string of letters, starting with the empty string, with the rule that there must be some uncapitalized word beginning with the given string of letters. (Thus you might start with q, but if so you would not be allowed to add x as the next letter.) The *loser* of the game is the first player to complete an uncapitalized word of at least three letters.

At first glance these mathematical games seem to have nothing in common, but on closer inspection one can notice that in all of them there are certain positions and rules telling you for each position which other positions you can move to. In Target, Nim, and Ghost, the same moves are available to both players (we call these games *impartial*), but this is not true in other games. For example, in chess, the white player can only move the white pieces while the black player can only move the black pieces. Moreover, you can always couch the rules of the game in such a way that the loser is the first player unable to move to a legal position. For example, in Target, the rule could be that you cannot make the total more than 28; in Ghost, it could be that you cannot add a letter to a string making it into either an uncapitalized word or another string such that no word begins with that string.

There is an abstract framework that can capture all such games at once. Calling the players of a game Left and Right, we decree inductively that if G_L and G_R are any two sets of games (including the empty set), then there is a game $G = (G_L|G_R)$; given such a game, if it is Left's turn, he moves to a game in G_L, which is itself an ordered pair of sets of games; if it is Right's turn, he moves to a game in G_R. All games (that is arise in this way. Note that we don't care what the positions are here, but only from which positions we can legally move to which others. The loser of a game is the first player unable to move.

What's the simplest possible game? Clearly, before we have any games at all, we have just one set of them, namely the empty set. Thus the first game is $G = (\emptyset|\emptyset)$; in it, by definition, neither player can move. We will see later that many games, including this one, have numbers attached to them called their values (though as promised in the title, these numbers can get pretty crazy). It shouldn't be too much of a stretch to guess that the number of G is 0; in fact, we identify G with the number 0.

The next two possibilities are $G_1 = (0|\emptyset)$ and $G_{-1} = (\emptyset|0)$; in both of them, one player has exactly one legal move (to 0) while the other player has no legal move. The values of these games are, unsurprisingly, 1 and -1, respectively. But consider now the game $(\emptyset|\emptyset)$, which we call $*$. By analogy with previous examples, the value of this game should be a number strictly between 0 and 0; but there is no such number. This is our first game with no number attached to it. We say that $*$ is *incomparable with 0*, that is, neither greater than, nor less than, nor equal to it. In general, a game with a positive number value is won by the Left player, regardless of who starts, while one with a negative number value is won by the Right player, regardless of who starts. A game with value 0 is won by the second player, whether Left or Right; a game like $*$ incomparable with 0 is won by the first player, again whether Left or Right. We haven't quite gotten yet to the "crazier numbers" of the title, as $*$ is too crazy even to be called a number, but we will see these numbers shortly.

The games $H = (0|1)$ and $K = (-1|0)$ should have values between 0 and 1 ad between -1 and 0, respectively; it shouldn't come as too much of a surprise that these values are respectively $1/2$ and $-1/2$. Now it is certainly easy to see that Left always wins H; but why should this game have the value $1/2$ rather than any other number between 0 and 1? The answer comes from a method of adding games: given games $G_1, \ldots, G_n$ we define their sum G by decreeing that the first player of G moves to a position P in one of the G_i, whereupon the second player moves either to a position from P or to a position in another G_i, and so on; the loser is the first player unable to move in any game. Now what happens when we add n copies of K to G_1? You guessed it: you can check that the second player wins, so that the sum has a value of 0, *exactly* when $n = 2$. For example, if $n = 2$, I am Left, and I make the first move, to -1 in the first copy of K, then my opponent moves to 0 in the second copy of K; now my only legal move is to 0 in G_1 and my opponent moves to 0 from the position -1 of K. Having no legal move I lose. Hence one plus twice the value of K equals 0, so that the value of K is $-1/2$, as claimed; similarly the value of H is $1/2$.

You might imagine that the value of $L = (0|1/2)$ would be $1/3$, since otherwise it's hard to see how any game could have value $1/3$. But in fact the value of L turns out to be $1/4$. In fact, it takes infinitely many steps to produce a game with value $1/3$, and such a game requires putting infinitely many numbers to the left or right (or both) of the vertical line. This is related to the infinite series $(1/4) + (1/16) + (1/64) + \ldots$, whose sum is $1/3$. But now we come to some more crazy numbers. Defining inductively the game $G_n = (G_0 \ldots G_{n-1}|\emptyset)$ for n a nonnegative integer, so that $G_0 = (\emptyset|\emptyset) = 0$, the game $G = (0, 1, \ldots |\emptyset)$ identifies with a number we call ω, which you can think of as ∞. Now we've all seen ∞ before, though it is often somewhat disrespected in the classroom, but wait until you see our next number $H = (0, 1, \ldots |\omega)$. This truly crazy number is called $\omega - 1$ and is quite distinct from ω! In a similar way we get $\omega - 1, \omega - 2$, and so on, all these numbers distinct from each other and from ω, and similarly $\omega + 1, \omega + 2$, and so on. We also get $\omega/2 = (0, 1, \ldots |\omega, \omega - 1, \ldots), \omega/4$, and so on, and eventually some really bizarre numbers like $\sqrt{\omega + 1}$. Don't talk about these numbers to your teachers, as you might go to jail for practicing fake mathematics, but be aware that they are out there.

As crazy as these new numbers are, it turns out that we can do all the usual arithmetic operations of addition, subtraction, multiplication, and division on them. For example ω has a multiplicative inverse, naturally denoted 1ω. We can even do more exotic things like taking nth roots for any n or finding roots of more complicated polynomial expressions like $x^5 - \omega x^2 + \sqrt{\omega + 1} = 0$. In defining these operations we need to rely on a very strong version of mathematical induction, according to which if we can define an operation on any two numbers coming from any of the sets S, T, U, V, and we are given two new numbers $x = (S|T)$ and $y = (U|V)$, then we can define our operation on x and y in terms of its values on numbers in S, T, U, V and thereby define it on any pair of numbers. I call this strong because while the ordinary positive integers form a clean and neat sequence $1, 2, 3, \ldots$, our numbers (which include all ordinary *real* numbers and many others) do not, so that a special type of induction, called *transfinite* is required; but it does the job.