Lecture 9-27: Linear algebraic groups

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I will cut right to the chase, beginning with an informal definition of linear algebraic groups, the main objects of study in this course, together with a number of examples. Shortly I will give a more formal definition and develop the concepts that logically preceded this definition. Throughout I will work over an algebraically closed field $\mathbf{k}$; I should mention that the second half of the Springer text together with many remarks in the first half are devoted to groups over arbitrary fields, but I will not need that level of generality here. All page references are to the Springer text "Linear algebraic groups", 2d edition.

Definition 2.1.1, p. 21

A linear algebraic group, often just called an algebraic group, over $\mathbf{k}$ is an affine variety $G \subset \mathbf{k}^n$ for some n (that is, the set of common zeros of some set T of polynomials in n variables over $\mathbf{k}$) which also has a group structure, such that multiplication and inversion on G are given by polynomial functions on it.

There is a clear similarity between this notion and that of Lie group (differentiable manifold with a group structure such that multiplication and inversion are given by smooth maps), but linear algebraic groups are more restricted than Lie groups, in three important ways.

- We are working with level sets (common zeros of a set of functions) rather than more general kinds of manifolds.
- We are restricting attention to polynomial functions rather than general smooth ones.
- We are not working over non-algebraically closed fields like $\mathbb{R}$.
- Linear algebraic groups are given to us by definition as subsets of $\mathbf{k}^n$ for some n; Lie groups are not given to us as subsets of Euclidean space.

Thus for example any linear algebraic group over the complex numbers $\mathbb{C}$ turns out to be a Lie group, but not conversely. Most of the standard examples of complex Lie groups that one sees in a first course on smooth manifolds do however turn out to be linear algebraic groups. A *nonexample* is $T = \{z \in \mathbb{C} : ||z|| = 1\}$; here the problem is that the condition $||z|| = 1$ for $z \in \mathbb{C}$ to lie in T is polynomial in the real and imaginary parts of z, but not in z itself.

In topology or geometry, T would be called a torus; but for this course the closest thing to T that is an algebraic group is $\mathbb{C}^*$, the multiplicative group of all nonzero complex numbers. This is what I will call a complex torus (or more generally any product of copies of $\mathbb{C}^*$).

Example

For any n the group $G = GL_n = GL_n(\mathbf{k})$ of $n \times n$ invertible matrices over $\mathbf{k}$ is linear algebraic. At first glance, this example does not seem to fit the definition, since although we can regard the elements of G as n^2-tuples $x = (x_1, \ldots, x_{n^2})$ over $\mathbf{k}$, the condition for such a tuple to lie in G is given by an inequality (nonvanishing of the determinant) rather than an equation.

Example

I get around this by adding a new variable y, defined to be the multiplicative inverse of the determinant $\det M$ of the matrix M corresponding to x whenever this determinant is nonzero. Then I can define G as the set of (n^2+1)-tuples $(x_1, \ldots, x_{n^2}, y)$ such that $(\det M)y = 1$. Then the condition to lie in G is polynomial. Moreover, the formula for the inverse of an invertible matrix, though not quite polynomial in its entries, *is* polynomial in these entries together with the inverse of its determinant, so the group operations on G are indeed polynomial in the coordinates.

Example

We also have $SL(n, \mathbf{k})$, the special linear group of $n \times n$ matrices over $\mathbf{k}$ of determinant 1; here we do not need any variables beyond the entries of the matrix. We further have the orthogonal group $O(n, \mathbf{k})$, consisting of all matrices M with $M^t = M^{-1}$ (where M^t is the transpose of M) and the special orthogonal group $SO(n, \mathbf{k}) = O(n, \mathbf{k}) \cap SL(n, \mathbf{k})$. If $n = 2m$ is even then we also have the symplectic group $Sp(n, \mathbf{k})$, defined by the equation $M^t = JM^{-1}J^{-1} = -JM^{-1}J$, where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and I is the $m \times m$ identity matrix; one checks easily that $Sp(n, \mathbf{k})$ is indeed closed under products, inverses, and transposes. For more examples see p. 23.

More generally, any closed subgroup of $GL(n, \mathbf{k})$, that is, one defined by polynomial equations, is a linear algebraic group. We have in particular the group D_n of invertible diagonal matrices, the group T_n of upper triangular invertible matrices, and the group U_n of unipotent upper triangular matrices (having ones on the main diagonal). Note that U_n is a normal subgroup of T_n; the quotient group T_n/U_n is naturally isomorphic to D_n. It turns out that in fact *all* linear algebraic groups arise as closed subgroups of $GL(n, \mathbf{k})$ for some n; this is why they are called linear. Thus for example $\mathbf{k}$ itself, regarded as an additive group, is linear algebraic; it is isomorphic to the matrix group $\{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbf{k}\}$. We also denote this group by G_a and call ti the additive group. Similarly $D_1 = GL_1$, consisting of the nonzero elements of $\mathbf{k}$, is an algebraic group under multiplication, often denoted G_m and called the multiplicative group.

Having exhibited some examples, it is now time to return to the beginning and proceed more formally.

Definition

As already mentioned above, an affine variety $V \subset \mathbf{k}^n$ is the set of $\mathcal{V}(T)$ of common zeros of some subset T of the polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$. The Zariski topology on V is the one for which the closed sets are the sets of common zeros in V of some subset T of S. In particular, $\mathbf{k}^n$ itself is an affine variety, often denoted $\mathbf{A}^n$.

Hilbert Nullstellensatz; Proposition 1.1.2, p. 1

There is a one-to-one inclusion-reversing correspondence from radical ideals I of S to affine varieties $V \subset \mathbf{k}^n$, sending the ideal I of S to $\mathcal{V}(I)$ and the variety V to the ideal $\mathcal{I}(V)$ of polynomials in S vanishing on V.

Definitions

Given an affine variety $V \subset \mathbf{k}^n$ its coordinate ring $\mathbf{k}[V]$ is the quotient $S/\mathcal{I}(V)$ of the polynomial ring S by the ideal of functions vanishing on V. If $W \subset \mathbf{k}^m$ is another affine variety with coordinate ring $\mathbf{k}[W]$ then a morphism ϕ from V to W is a the restriction to V of a polynomial map from $\mathbf{k}^n$ to $\mathbf{k}^m$ mapping V to W, where we identify any two such maps if they have the same restriction to V.

A morphism $\phi : V \to W$ identifies naturally with a $\mathbf{k}$-algebra homomorphism $\phi^* : k[W] \to k[V]$, since any such $\mathbf{k}$-algebra homomorphism is determined by where it sends the generators of $\mathbf{k}[x_1, \ldots, x_m]$. I will develop the many connections between ϕ and ϕ^* later.