Lecture 11-15: More about root systems

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We further develop the theory of root systems, defining the Dynkin diagram of such systems and showing that the systems can be recovered from their diagrams. Some of the material appears in Chapter 9 of the text, but we will go beyond what is given there.

Let $R \subset V$ be a root system, R^+ a positive subsystem, and Δ the corresponding simple subsystem. We first claim that $(\alpha, \beta) \leq 0$ *for* $\alpha, \beta \in \Delta$. Otherwise Cauchy-Schwarz forces one of $(\alpha, \check{\beta}), (\beta, \check{\alpha})$ to equal 1, whence $\alpha - \beta, \beta - \alpha$ are roots (equal to $\pm s_\alpha(\beta)$ or $\pm s_\beta(\alpha)$), one of which must be positive. Then $\alpha = \alpha - \beta + \beta$ or $\beta = \beta - \alpha + \alpha$ is not indecomposable, a contradiction. Next we claim that *the simple roots are linearly independent*. Indeed, there first of all cannot be a nontrivial dependence relation $\sum_{\alpha \in \Delta} n_\alpha \alpha = 0$ with the $n_\alpha \in \mathbb{R}$ nonnegative, for then the dot product of the vector $x \in V$ corresponding to R^+ and the left side would be positive. Next, we cannot have a relation $\sum_{\alpha \in \Delta_1 \subset \Delta} n_\alpha \alpha = \sum_{\alpha \in \Delta - \Delta_1} m_\alpha \alpha$ with the n_α, m_α nonnegative and Δ_1 a proper subset of Δ, for then the dot product of the left side and some $\alpha \in \Delta_1$ would have to be positive, while the dot product of the right side and α is nonpositive. The claim follows.

Finally, *any positive root* β *is conjugate by a product of reflections* s_α *corresponding to simple roots* α *(simple reflections) to a simple root*. Indeed, writing $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ as a nonnegative integral combination of simple roots α, we must have $(\beta, \alpha) > 0$ for some $\alpha \in \Delta$, whence $s_\alpha \beta$ is a root, which must be positive if $\beta \neq \alpha$, since it involves a simple root different from α with positive coefficient. Iterating this process, we find that some product of simple reflections sends β to a simple root. Similarly, any negative root, being the negative of a positive root, is also conjugate by a product of simple reflections to a simple root. Hence all roots are obtained by repeatedly applying simple reflections to simple roots, and if we know the ratio $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ for any simple roots α, β, then we know how to compute any product of simple reflections applied to a simple root, and so can recover R completely.

It is convenient to encode the ratios $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ as follows. Construct a graph in which the nodes are indexed by the simple roots. The nodes corresponding to the roots α, β are then joined by $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$ edges, with an additional arrow pointing to the shorter of α, β if these vectors do not have the same length. (From this information we can compute the roots $s_\alpha(\beta), s_\beta(\alpha)$.) In this way we get the *Dynkin diagram* of R (see p. 168). We now compute the root systems and Dynkin diagrams corresponding to the algebraic groups $SL_n(\mathbf{k}), SO_n(\mathbf{k})$, and $Sp(2n, \mathbf{k})$ defined previously in the course. What we actually do is compute the root spaces in the Lie algebra, which are easier to describe explicitly; these are the eigenspaces of the adjoint action of a maximal torus T on this algebra. In all cases T will consist of diagonal matrices and the adjoint action is given by conjugation.

We first take $G = SL_\ell(\mathbf{k})$; we choose this group rather than $GL_\ell(\mathbf{k})$ because the latter group is reductive but not semisimple. Here T is the subgroup of D_n consisting of diagonal matrices of determinant one. For every index i, the character of T sending a diagonal matrix $t =$diag$(t_1, \ldots, t_\ell)$ to t_i will be denoted e_i, so that if $i \neq j$ the character sending t to $t_i t_j^{-1}$ is denoted $e_i - e_j$. Then T acts on the *root space* $\mathfrak{g}_{ij} = \mathbf{k} e_{ij}$ spanned by the matrix unit e_{ij} (having a 1 in the *ij*th entry and zeroes elsewhere) by the character $e_i - e_j$ and the Lie algebra $\mathfrak{g}$ is spanned by the Lie algebra $\mathfrak{t}$ of T together with the $\mathfrak{g}_{ij}$. Accordingly the corresponding root system consists of differences $e_i - e_j \in \mathbb{R}^\ell$ of unit coordinate vectors in $\mathbb{R}^\ell$.

As a set of positive roots we take the differences $e_i - e_j$ with $i < j$; as the corresponding set of simple roots we then have $e_1 - e_2, \ldots, e_{\ell-1} - e_\ell$. The Dynkin diagram thus consists of a single chain of $\ell - 1$ dots, each connected to its neighbors by single edges. This diagram is said to be of *type* $A_{\ell-1}$; the subscript is $\ell - 1$ rather than *ell* since the rank is $\ell - 1$. The dimension of G is $\ell^2 - 1$. The Weyl group W is isomorphic to S_ℓ, the group of permutations of the coordinates of $\mathbf{k}^\ell$.

Next take $G = Sp_{2\ell}(\mathbf{k})$. Setting $s = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I denotes the $\ell \times \ell$ identity matrix, we can identify G with the set of matrices M such that $M^t s M = s$ and then $\mathfrak{g}$ is identified with the set of matrices $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ such that $sx = -x^t s$, or equivalently $n^t = n, p^t = p, m^t = -q$, where m, n, p, q are $\ell \times \ell$ matrices. Here T consists of all matrices of the form $t =$diag$(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1})$; we have characters $e_i - e_j$ as above for all indices i, j with $i \neq j, 1 \leq i, j \leq \ell$ and in addition $2e_i$, sending t to t_i^2, and $e_i + e_j$, sending t to $t_i t_j$ (again for $1 \leq i, j \leq \ell$).

The Lie algebra is spanned by the diagonal matrices $e_{ii} - e_{\ell+i,\ell+i}$ in it together with all differences $e_{ij} - e_{\ell+j,\ell+i}$, units $e_{i,\ell+i}$ and $e_{\ell+i,i}$, and sums $e_{i,\ell+j} + e_{j,\ell+i}$ and $e_{\ell+i,j} + e_{\ell+j,i}$, for $1 \leq i \neq j \leq \ell$. The differences, sums, and units all span root spaces in $\mathfrak{g}$; the corresponding roots are $e_i - e_j, \pm 2e_i$, and $\pm(e_i + e_j)$ for $1 \leq i \neq j \leq \ell$. These vectors thus form a root system, said to be of *type* C_ℓ. As positive roots we take the vectors $e_i + e_j, 2e_i$, and $e_i - e_j$ with $i < j$; the corresponding simple roots are $e_1 - e_2, \ldots, e_{\ell-1} - e_\ell, 2e_\ell$. The Dynkin diagram consists of a chain of ℓ dots arranged as for type A_ℓ, except that the rightmost dot is connected to its neighbor by a double edge together with an arrow pointing to the left. The dimension of G is $2\ell^2 + \ell$. The Weyl group is the hyperoctahedral group H_ℓ consisting of all permutations and sign changes of the coordinates of $\mathbf{k}^\ell$ (or the symmetry group of the hypercube in $\mathbf{k}^\ell$ having vertices $(\pm 1, \ldots, \pm 1)$).

Next we take $G = SO_{2\ell+1}(\mathbf{k})$; this case behaves sufficiently differently from the case of $SO_{2\ell}(\mathbf{k})$ to warrant inclusion in a different category. We need to realize G differently than we did before; instead of consisting of all matrices M with $M^t M = I$, we define the matrix $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$ (with I denoting the $\ell \times \ell$ identity matrix) and take G to consist of all matrices M with $M^t sM = s, \det M = 1$. We realize G in this way so that we can take T to be the diagonal matrices in G; in the previous realization there are only finitely many diagonal matrices in G.

As in type C_ℓ the Lie algebra $\mathfrak{g}$ consists of all matrices x with $sx = -x^t s$; writing s as $\begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$, where $a \in \mathbf{k}, b_1, b_2$ are row vectors, c_1, c_2 are column vectors, both in $\mathbf{k}^\ell$, and m, n, p, q are $\ell \times \ell$ matrices, this condition translates to the conditions $a = 0, c_1 = -b_2^t, c_2 = -b_1^t, q = -m^t n^t = -n, p^t = -p$. The torus T consists of diagonal matrices of the form $\mathrm{diag}(1, t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1})$; we define characters $\pm e_i$ and $\pm(e_i \pm e_j)$ as in type C.

In this case the root spaces in $\mathfrak{g}$ are spanned by the differences $e_{1,\ell+i+1} - e_{i+1,1}, e_{1,i+1} - e_{\ell+i+1,1} (1 \leq i \leq \ell), e_{i+1,j+1} - e_{\ell+j+1,\ell+i+1} (1 \leq i \neq j \leq \ell), e_{i+1,\ell+j+1} - e_{j+1,\ell+i+1} (1 \leq i < j \leq \ell)$, and $e_{i+\ell+1,j+1} - e_{j+\ell+1,i+1} (1 \leq j < i \leq \ell)$. The roots are $\pm e_i$ and $\pm(e_i \pm e_j)$, forming a system of *type* B_ℓ; note that these roots differ from those in type C_ℓ only in that $2e_i$ is replaced by e_i for all i. As positive roots we take the $e_i, e_i - e_j$, and $e_i + e_j$ for $i < j$; as simple roots we then get $e_1 - e_2, \ldots, e_{\ell-1} - e_\ell, e_\ell$. The Dynkin diagram consists of a chain of ℓ dots, with the rightmost dot connected to its neighbor by a double edge together with an arrow, this time pointing to the right. The dimension $2\ell^2 + \ell$ of G is the same as it was in type C_ℓ. The Weyl group is also the same as it is for type C_ℓ. The groups $Sp_{2\ell}(\mathbf{k})$ and $SO_{2\ell+1}(\mathbf{k})$ are *dual* to each other in the sense that if $(X, R, \check{X}, \check{R})$ is the root datum of one of them, then $(\check{X}, \check{R}, X, R)$ is the root datum of the other.

Finally we take $G = SO_{2\ell}(\mathbf{k})$. Realizing G as in type B_ℓ, but this time taking $s = \begin{pmatrix} 0 & I \\ i & 0 \end{pmatrix}$, we find by a computation almost identical to that in type B_ℓ, but without the extra complication of the first row and column, that the roots are $\pm(e_i \pm e_j)$ for $1 \leq i < j \leq \ell$. As positive roots we take $e_i - e_j, e_i + e_j$ for $i < j$; then the simple roots become $e_1 - e_2, \ldots, e_{\ell-2} - e_{\ell-1}, e_{\ell-1} \pm e_\ell$. The Dynkin diagram consists of a chain of $\ell - 2$ dots together with two additional unconnected neighbors of the rightmost dot, each connected to it by a single edge. The dimension of G is $2\ell^2 - \ell$; the type of the root system, or of G, is (can you guess?) D_ℓ. The Weyl group does not have a name, but it is the subgroup of H_ℓ of index 2 consisting of the coordinate permutations and sign changes involving evenly many signs. The types A_ℓ through D_ℓ, together with the corresponding groups and Lie algebras, are called *classical*; the remaining types are called *exceptional*.

It turns out that just five additional connected Dynkin diagrams arise, having the labels E_6, E_7, E_8, F_4, and G_2; note that we have already constructed the root system G_2 this week. We can construct all five root systems in a uniform manner, as follows. In each case, we start with a specified lattice ($\mathbb{Z}$-span of an $\mathbb{R}$-basis of a real vector space) and then take the vectors in it of one or two specified lengths. The axioms of a root system are then easy to verify in all cases.

To construct the systems of type E, start with the lattice $L \subset \mathbb{R}^8$ consisting of all $a_1, \ldots, a_8$) such that either all $a_i \in \mathbb{Z}$ or all $a_i \in \mathbb{Z} + \frac{1}{2}$ and in addition $\sum a_i \in 2\mathbb{Z}$. To construct E_8 we take the set of all vectors of square length 2 in L; it turns out that there are exactly 240 such vectors. They consist of all $\pm e_i \pm e_j$ with $i > j$ together with $\{(1/2)(s_1, \ldots, s_8) : s_i = \pm 1, \prod s_i = 1\}$. As positive roots we take $e_i \pm e_j$ with $i > j$ (note the different convention in this case) together with the $(1/2)(s_1, \ldots, s_8)$ with $s_8 = 1$. The simple roots are then
$(1/2)(1, -1, \ldots, -1, 1), e_2 - e_1, e_2 + e_1, e_3 - e_2, e_4 - e_3, \ldots, e_7 - e_6$.
The Dynkin diagram, depicted on p. 168, consists of a chain of seven dots together with an additional neighbor of the third dot, counting from the left end. To construct E_7, take the roots in E_8 orthogonal to $e_7 + e_8$; to construct E_6, take the roots in E_8 orthogonal to both $e_7 + e_8$ and $e_6 + e_8$. Their diagrams are obtained from that of E_8 by chopping off the one or two rightmost dots.

For F_4 take L to be the lattice in $\mathbb{R}^4$ spanned by the unit coordinate vectors e_i together with the vector $(1/2)(1,1,1,1)$. To get the root system, take all vectors of square length 1 or 2 in L, obtaining thereby all $\pm e_i, \pm e_i \pm e_j$, and all $(1/2)(s_1, s_2, s_3, s_4)$, where the signs s_i may be chosen arbitrarily. As positive roots we take the $e_i, e_i + e_j, e_i - e_j$ for $i < j$, and the $(1/2)(s_1, s_2, s_3, s_4)$ with $s_1 = 1$. The simple roots are then $e_2 - e_3, e_3 - e_4, e_4$, and $(1/2)(1, -1, -1, -1)$; the diagram consists of a chain of four dots whose middle link is a double edge, with the arrow pointing in either direction. Finally, for G_2, as noted previously, start with the lattice L spanned by $e_1 - e_2$ and $e_2 - e_3$ in $\mathbb{R}^3$ and take all vectors of square length 2 or 6 in L. The simple roots may be taken to be $(-2, 1, 1), (0, 1, -1)$; the diagram is a pair of dots connected by a triple edge, with the arrow again pointing in either direction. The Weyl group of type G_2 is the direct product $S_3 \times \mathbb{Z}_2$; here the cyclic factor $\mathbb{Z}_2$ acts on $\mathbf{k}^3$ by the scalar -1.