Lecture 10-6: First results on algebraic groups

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With the basic algebraic geometry that I need in place, I am ready to take the study of algebraic groups, following Chapter 2. Throughout G denotes a (linear) algebraic group.

Proposition 2.2.1, p. 25

The connected component G^0 of G containing the identity e (called its identity component) is also the unique irreducible component of G containing the identity. It is a closed normal subgroup of finite index and any closed normal subgroup of finite index contains it.

Proof.

If X and Y are irreducible components of G containing e then the product XY, being the closure of the product $X \times Y$ under the multiplication map, is also irreducible; likewise its closure $\overline{XY}$ is irreducible, forcing $X = Y = \overline{XY}$. Also X^{-1}, being the image of X under the inverse map, is irreducible, forcing $X = X^{-1}$, whence X is a closed subgroup. Since inner automorphisms define homeomorphisms of G, any conjugate xXx^{-1} of X coincides with X, so X is the unique irreducible component of G containing e and is a normal subgroup. The cosets xX of X are also irreducible components, whence there are only finitely many of them and $X = G^0$, since the irreducible components of G are disjoint. If H is a closed normal subgroup of G with finite index, then H^0 is closed and of finite index in G^0, forcing $H^0 = G^0$, so that H contains G^0. □

In particular, G is connected if and only if it is irreducible; it is customary to speak of connected algebraic groups rather than irreducible ones.

Lemma 2.2.4, p. 26

If H is a subgroup of G then its closure $\overline{H}$ is also a subgroup of G; if H contains an open subset of $\overline{H}$ then H is already closed.

Proof.

Note first that if U, V are dense open subsets of G and $x \in G$, then U must meet the dense open subset xV^{-1} of G, whence $UV = G$. Now let $x \in H$. Then $H = xH \subset x\overline{H}$. Since $x\overline{H}$ is closed we have $x\overline{H} \subset \overline{H}$ and $H\overline{H} \subset \overline{H}$. Similar arguments show that $\overline{H}$ is closed under multiplication and inversion, so $\overline{H}$ is a subgroup. If $U \subset H$ is open in $\overline{H}$ and nonempty, then H, being a union of translates of U, is open in $\overline{H}$, whence $\overline{H} = H.H = H$. $\square$

Proposition 2.2.5, p. 26

Let $\phi : G \to G'$ be a homomorphism of algebraic groups. Then $\ker \phi$ is a closed normal subgroup of G, the image $\phi(G)$ is a closed subgroup of G', and $\phi(G^0) = (\phi G)^0$.

Proof.

The kernel $\phi^{-1}(e)$ is closed in G and we know that the image $\phi(G)$ contains a nonempty subset of its closure, whence it must be closed. Finally $\phi(G^0)$ is connected and of finite index in $(\phi G)^0$, whence it coincides with the latter. $\square$

Proposition 2.2.6, p. 26

Let $(X_i, \phi_i)_{i\in I}$ be a family of irreducible varieties together with morphisms $\phi_i : X_i \to G$. Let H be the smallest closed subgroup of G containing the images $Y_i = \phi_i(X_i)$; assume that all Y_i contain the identity element e. Then H is connected. There exists a positive integer n and $a(1), \ldots a(n) \in I$ and for each i an exponent $\epsilon(i) = \pm 1$ such that $H = Y_{a(1)}^{\epsilon(1)} \ldots Y_{a(n)}^{\epsilon(n)}$.

Proof.

Enlarging the index set I if necessary, we may assume that all inverses Y_i^{-1} occur among the Y_i. For every n-tuple $a = (a(1), \ldots, a_n) \in I^n$ write $Y_a = Y_{a(1)} \ldots Y_{a(n)}$; then every Y_a and its closure $\overline{Y_a}$ is irreducible; using an obvious notation we can write $Y_b.Y_c \subset Y_{(b,c)}$. Arguing as in a previous result, we get $\overline{Y_b}.\overline{Y_c} \subset \overline{Y_{(b,c)}}$. Choosing a such that $\dim Y_a$ is maximal we deduce that $\overline{Y_a}$ is a subgroup and $\overline{Y_a} = Y_a.Y_a$. Then $H = Y_a$ has the desired properties. □

This result, together with some basic linear algebra, is very useful in showing that some of the algebraic groups introduced during the first week are connected. It also forms the basis of the construction we will give later of a reductive group from something called a root datum.

Corollary 2.27, p. 27

Given a family $(G_i)_{i\in I}$ of closed connected subgroups of a group G, the subgroup H that they generate is closed and connected. Also there are $a(1),\ldots,a_n \in I$ such that $H = G_{a(1)}\ldots G_{a(n)}$.

Corollary 2.28, p. 27

If H, K are subgroups of G with at least one of them connected, then the subgroup (H, K) generated (by definition) by all commutators $hkh^{-1}k^{-1}$ with $h \in H, k \in K$, is connected. In particular the commutator subgroup (G, G) of G is connected whenever G is.

I conclude with a discussion of G-spaces.

Definition 2.3.1, p. 28

Given an algebraic group G and a variety X we say that X is a *G-space* if G acts on X by morphisms, so that there is a morphism $a : G \times X \to X$ of varieties such that $a(g, a(h, x)) = a(gh, x), a(e, x) = x$. Write $g.x$ for $a(g, x)$. As usual we say that X is homogeneous if it has just one orbit under the action of G (so that for any $x \in X$ the set $G.x = \{g.x : g \in G\}$ is all of X). Given a G-space X the comorphism a^* of the action map $a : G \times X \to X$ maps $\mathbf{k}[X]$ to $\mathbf{k}[G \times X] = \mathbf{k}[G] \otimes \mathbf{k}[X]$.

In particular, if V is a $\mathbf{k}$-vector space on which G acts by linear transformations, so that $g.(x+y) = g.x + g.y, g.(kx) = kg.x$ for $k \in \mathbf{k}, x, y \in V$, then we call V a *representation* of G. If V has finite dimension n, then it is equivalent to require that there be a homomorphism $r : G \to GL(n, \mathbf{k})$ of algebraic groups. In this case we say that the representation is *rational*. If $n = 1$ we call r a *character* of G. Given a G-space X we get a representation of G on $\mathbf{k}[X]$ via the recipe $(g.f)(x) = f(g^{-1}x)$ for $g \in G, x \in X$.