Mathday Talk–2018

I will begin with one of the problems mentioned in the abstract. We are given a league of n soccer teams who play each other at various points throughout the season, the only constraint on the schedule being that no team plays any other more than once. We must show that at *every* point during the season, some pair of teams have played the same number of games (possibly 0). What makes this problem hard is that we know nothing about the schedule apart from what has been indicated above; it seems that we would have to consider infinitely many possible schedules to prove this result.

I solve this problem using graph theory, the subject of the talk. Here I must hasten to add that I am not talking about the graphs you have learned about in school; my graphs have nothing to do with Cartesian coordinates or functions. Instead, a graph for me consists by definition of a finite set of points in the plane, called vertices, with some but not necessarily all pairs of them joined by curves or straight lines. If two of these curves happen to intersect at a point other than a vertex, then the intersection point does *not* count as a new vertex; instead it is just ignored. The *degree* of every vertex is the number of edges coming out of it.

It is not difficult to see how the soccer problem can be modelled by a graph in this sense. The vertices are just the teams in the league. At every point in the season we join two vertices by an edge if and only if the corresponding teams have played each other by then. Now I must show that two vertices have the same degree. Since there are n teams in the league and no team plays any other more than once, the possible vertex degrees are $0, 1, \ldots, n-1$. At first it seems possible that all of these numbers might occur as vertex degrees; but on reflection we note that if the degree $n-1$ occurs, then some team has played every other, so that no team has played 0 others. Hence at most $n-1$ distinct numbers can occur as vertex degrees, and some two of the n vertices have the same degree, as desired. (As an exercise show that it is indeed possible for the vertex degrees to be $0, \ldots, n-1$ in some order.)

As promised in the title, I now consider a totally different problem. Suppose that six people are in a room. Every pair of them are either acquaintances or strangers; we assume that if A knows B, then B also knows A. The problem is to show that some three people in the room are either mutual acquaintances (so that each knows the other two) or mutual strangers (so that none of them knows either of the others). Here I set up a graph with six vertices, one for each person, but now we need to modify the above definition of graph slightly to keep track of who is acquainted with whom: I join *every* pair of distinct vertices by an edge but color the edges either red or black, according as the people are acquaintances or strangers. I call such a graph a *complete bichromatic graph* (complete because every pair of vertices is joined by an edge, bichromatic because the edges now come in two different colors). In these terms, our problem is now to show that every complete bichromatic graph on six vertices has a monochromatic triangle, that is, a set of three vertices such that the edges joining any two of them have the same color. I do this by considering the five edges coming out of any one vertex. Some three of these must have the same color, say red. Now look at the three vertices joined to the given one by red edges. If any two are joined by a red edge, then we have a red triangle; if all of them are joined by black edges, then we have a black triangle. Thus we must have a monochromatic triangle, one way of the other.

In fact, arguing in a different way, we can prove a stronger result. For each vertex, count the number of pairs of edges coming out of it with *different* colors. The number of such pairs of edges is either 0,4, or 6, according as all such edges have the same color, one has one color and four the other, or two edges have one color and the other three the other. Adding over all vertices, we see that the number of such pairs of edges is at most 36; it is exactly 36 if and only if every vertex has three edges coming out of it with one color and two with the other. On the other hand, every such pair of edges lies in a unique *bichromatic* triangle, which furnishes exactly two such pairs of edges. The total number of triangles of vertices is 20: we have 6 choices for the first vertex, 5 for the second, and 4 for the third; but then the three vertices of any triangle can be listed in any of 6 possible orders, whence the total number of triangles is $(6\cdot5\cdot4)/6 = 20$. Hence there can be at most 18 bichromatic triangles and there must be at least 2 (not just one) monochromatic one. There are essentially just two different ways to get exactly two monochromatic triangles; either start with a pair of disjoint black triangles and join every vertex of one of them to every vertex of the other by a red edge, or start with a complete bichromatic graph on five vertices with no monochromatic triangle (as an exercise, find such a graph) and join a sixth vertex to the other five with two red edges and three black ones. No matter how this is done, you will find that you get exactly two monochromatic triangles, both sharing the sixth vertex; they may have the same or different colors.