Lecture 9-29: Zariski topology and regular functions

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Let $X \subset \mathbf{k}^n$ be an affine variety with coordinate ring $\mathbf{k}[X]$. Last time I reviewed the Zariski topology on X, in which the closed subsets are the zero sets $\mathcal{V}(I)$ of (radical) ideals I of $\mathbf{k}[X]$. In the special case $\mathbf{k} = \mathbb{C}$, this topology differs from the inherited Euclidean topology of X in two crucial ways: there are many fewer open sets and the nonempty ones are much fatter. More precisely, any open set is a finite union of basic or principal open sets $D(f)$, where $D(f)$ is the set of nonzeros of a single polynomial $f \in \mathbf{k}[X]$. Any such open set $D(f) \subset \mathbf{k}^n$ has the structure of an affine variety, since it is isomorphic to the set $\{(x_1, \ldots, x_n, y) \in \mathbf{k}^{n+1} : f(x_1, \ldots, x_n)y = 1\}$. Since there are no infinite ascending chains of ideals in $\mathbf{k}[X]$, there are no infinite descending chains of closed sets and no infinite ascending chains of open sets. In particular, X is quasicompact as a topological space; I say "quasicompact" rather than "compact" because X is almost never a Hausdorff space.

We call a general topological space X irreducible if it is not the union of two proper closed subsets, or equivalently if every nonempty open subset of X is dense. If X is an affine variety, then this holds if and only if the ideal I=$I \subset \mathbf{k}[x_1, \ldots, x_n]$ corresponding to X is prime, or if and only if the coordinate ring $\mathbf{k}[X]$ is an integral domain. In this case the coordinate ring $\mathbf{k}[X]$ has a quotient field $\mathbf{k}(X)$, whose transcendence degree over $\mathbf{k}$ is defined to be the dimension $\dim X$ of X (see 1.8.1, p. 16). In general an affine variety X can be uniquely written as a finite union $\cup_{i=1}^m X_i$ of irreducible subsets X_i such that no X_i properly contains another; these are called its *irreducible components*. The maximum dimension of the X_i is defined to be the dimension of X in this case. A proper closed subvariety of an irreducible affine variety has smaller dimension (see Proposition 1.8.2 on p. 19).

I need to work with a larger class of functions than the polynomial ones; specifically, I need to consider certain quotients of polynomial functions. Call a $\mathbf{k}$-valued function f defined on a nonempty open subset U of X regular if for every $x \in U$ there is a neighborhood V of x and $g_x, h_x \in \mathbf{k}[X]$ such that h is never 0 on V and $f = g/h$ there (1.4.1, p. 6). For each nonempty open subset U of X I denote by $\mathcal{O}_U = \mathcal{O}_U(X)$ the $\mathbf{k}$-algebra of regular functions defined on U.

These algebras satisfy the obvious properties that if $V \subset U$ is open and nonempty, then restriction of functions defines a **k**-algebra homomorphism $\mathcal{O}_U \to \mathcal{O}_V$; and if $\cup_{\alpha\in A} U_\alpha$ is an open covering of U by sets U_α and we are given regular functions $f_\alpha \in \mathcal{O}_{U_\alpha}$ for each $\alpha \in A$ such that f_α, f_β agree on the intersection $U_\alpha \cap U_\beta$, then there is a unique $f \in \mathcal{O}_U$ such that f restricts to f_α on U_α for all α (see p. 6 of Springer).

It is not difficult to show that a regular function defined on all of X is just an element of the coordinate ring $\mathbf{k}[X]$ (see Theorem 1.4.5 on p. 8). If U is a proper open subset of X, then there can however be regular functions on U not defined on all of X, e.g. the function $1/x$ is defined at all points of $\mathbf{k}$ except $x = 0$. More generally, any topological space X together with a collection $\mathcal{O}_X$ of rings $\mathcal{O}_U$ of $\mathbf{k}$-valued functions on U, one for each nonempty open subset U of X and satisfying the properties of the last slide, is called a ringed space. If X is a ringed space and U is a nonempty open subset, then the induced ringed space $(U, \mathcal{O}_X|U)$ is attached to the collection of rings $\mathcal{O}_V$, as V runs through the open subsets of U. Then I have

Definition; see p. 9

A morphism $\phi : X \to Y$ from (the ringed spaces $\mathcal{O}_X, \mathcal{O}_Y$ corresponding to) X to Y is a continuous map from X to Y such that for any open subset U of Y the induced map ϕ^* on $\mathcal{O}_U(Y)$ maps it into $\mathcal{O}_{\phi^{-1}(U)}X$.

One easily checks that a morphism $\phi : X \to Y$ in this sense between a pair of affine varieties X, Y is just a polynomial map between the ambient spaces $\mathbf{k}^n, \mathbf{k}^m$ containing X, Y and mapping X into Y, as it was last time. There is a bijective correspondence between such morphisms ϕ and $\mathbf{k}$-algebra homomorphisms $\phi^* : \mathbf{k}[Y] \to \mathbf{k}[X]$.

I will conclude with some simple relations between properties of a morphism ϕ and the homomorphism ϕ^*.

Lemma 1.9.1 on p. 17

Given a morphism $\phi : X \to Y$ between affine varieties X, Y and the corresponding homomorphism $\phi^* : \mathbf{k}[Y] \to \mathbf{k}[X]$, we have

- if ϕ^* is onto then ϕ maps X onto a closed subvariety of Y.
- ϕ^* is one-to-one if and only if the closure $\overline{\phi X}$ of ϕX equals Y.
- if X is irreducible then so is $\overline{\phi X}$ and $\dim \overline{\phi X} \leq \dim X$.

Proof.

Let $I = \ker \phi^*$. If ϕ^* is onto then ϕX is the variety $\mathcal{V}_Y(I)$ corresponding to I, whence the first assertion. Also ϕ^* is one-to-one if and only if the ideal $\mathcal{I}_Y(\phi X)$ corresponding to $\phi X \subset Y$ is 0, whence the second assertion. The third assertion is a general fact about continuous maps from one irreducible topological space to another, together with the previous fact that $\mathbf{k}[X]$ is a quotient of $\mathbf{k}[\overline{\phi X}]$ in this situation. □

Note that the image ϕX need not be closed, even if it is dense in Y; as an example, take X to be $\{(x, y) \in \mathbf{k}^2 : xy = 1\}$, $Y = \mathbf{k}$, $\phi(x, y) = x$. I will show next time that ϕX need not be open in its closure either.