Lecture 12-8: Chevalley groups and real Lie groups

December 8, 2023

In this last lecture we generalize the construction of adjoint groups from root systems beyond the simply laced case. The construction works over an arbitrary basefield **k** and gives rise to the Chevalley groups mentioned in the lecture on December 1. We also show how to construct certain real Lie (but non-algebraic) groups and Lie algebras from such systems, starting from the complex ones already constructed.

Given a root system R, our first task is to construct a Lie algebra with this root system; we have already seen how to do this in the simply laced case. In general, we define $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in R} \mathbf{k} e_\alpha$ as a$\mathbf{k}$-vector space exactly as before. Once again we decree that $[u.u'] = 0, [u, e_\alpha] = \langle \alpha, u \rangle e_\alpha, [e_\alpha, e_{-\alpha}] = 1 \otimes \check{\alpha}$ for $u, u' \in \mathfrak{t}, \alpha \in R$. We also set $[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in R$. It remains to define the structure constants $c_{\alpha,\beta}$ so that the Jacobi identity holds. A well-known result of Chevalley asserts that this can always be done and we can arrange that $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$; whenever this last property holds we automatically have $c_{\alpha,\beta} = \pm(r+1)$, where r iis the largest nonnegative integer such that $\beta - r\alpha \in R$ (cf. Proposition 9.5.3). This follows from a more elaborate version of the construction given above in the simply laced case; or else we can give an abstract presentation of $\mathfrak{g}$ by generators and relations (following Serre) and then use an automorphism of $\mathfrak{g}$ of order 2 to find root vectors e_α with the desired properties.

The formula for the constant $c_{\alpha,\beta}$ up to sign shows (after a brief calculation) that the coefficients of all powers of k in the series for $\exp \operatorname{ad} ke_\alpha(e_\beta)$ are binomial coefficients, up to sign, for any $k \in \mathbf{k}$ and $\alpha, \beta \in R$; thanks to the nilpotence of $\operatorname{ad} ke_\alpha$, the series for $\exp \operatorname{ad} ke_\alpha$ thus makes sense on $\mathfrak{g}$ for any $k \in \mathbf{k}$, regardless of the characteristic. Then one can form the group of automorphisms of $\mathfrak{g}$ generated by the torus T with character group $\mathbb{Z}R$, together with the groups U_α for $\alpha \in R$ defined as before. This is called the *Chevalley group* attached to R and $\mathbf{k}$. Chevalley showed that if R is irreducible then this group is such that every proper normal subgroup is central, apart from a few exceptions for very small $\mathbf{k}$. By proving this, he was able to exhibit several new families of previously unknown finite simple groups. In fact, the alternating groups, the finite Chevalley groups, and modifications of these groups called twisted Chevalley groups turn out to account for all finite simple groups with just 26 exceptions (the so-called *sporadic* simple groups).

The Lie algebra $\mathfrak{g}$ constructed as above is called *split*; if R is irreducible and the characteristic of $\mathbf{k}$ is either 0 or large enough, then $\mathfrak{g}$ is *simple* in the sense that it has no nonzero proper ideals $\mathfrak{h}$, that is, no proper subspaces $\mathfrak{h}$ such that $[x, \mathfrak{h}] \subset \mathfrak{h}$ for $x \in \mathfrak{g}$.

We conclude with a construction of a very different class of Lie algebras and adjoint groups. Given the root system R, let D be a choice of simple roots. Let $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in R} \mathbb{C} e_\alpha$ be constructed as above, using the basefield $\mathbb{C}$ of complex numbers. Fix a square root i of -1 and let $\mathfrak{k} = \mathfrak{g}_\mathbb{R}$ be the *real* span of the $i\check{\alpha}$ and the vectors $e_\beta - e_{-\beta}, i(e_\beta + e_{-\beta})$ as α, β run over D and the corresponding set R^+ of positive roots, respectively, Since the structure constants $c_{\gamma,\delta}$ are real and satisfy $c_{\gamma,\delta} = -c_{-\gamma,-\delta}$, one checks immediately that $\mathfrak{k}$ is closed under the Lie bracket. There is however no analogue for $\mathfrak{k}$ of the root spaces $\mathfrak{g}_\alpha$. In fact it turns out for any $x \in \mathfrak{k}$ that ad x acts on the complexification $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ of $\mathfrak{k}$ with purely imaginary eigenvalues, so that even though ad x acts semisimply on $\mathfrak{k}$, it does not act diagonally, unless $x = 0$.

Given $\mathfrak{k}$, we now define its *adjoint group* K much as we did before, taking this group to be generated by the exp ad $ki\check{\alpha}$, exp ad $k(e_\beta - e_{-\beta})$, and exp ad $ki(e_\beta + e_{-\beta})$ as k runs over $\mathbb{R}, \alpha$ over D, and β over R^+. This time however the elements $x \in \mathfrak{k}$ involved do *not* have ad x nilpotent, so we need the infinite series definition of exp ad x and the completeness of the real field to define K. To see that this group is compact, it is helpful to introduce an analogue for general Lie algebras $\mathfrak{h}$ (over any basefield $\mathbf{k}$) of the pairing $\langle \cdot, \cdot \rangle$ appearing in the definition of root datum and the associated bilinear form $(\cdot, \cdot)$.

The *Killing form* κ on $\mathfrak{h}$ is defined by $\kappa(x, y) =$tr(ad x)(ad y), the trace of the product of the linear transformations ad x and ad y on $\mathfrak{h}$. This is clearly a symmetric bilinear form; for basefields **k** of characteristic 0 it turns out that this form is nondegenerate if and only if $\mathfrak{h}$ is *semisimple* in the sense of having no nonzero solvable ideals. Here solvability is defined for Lie algebras analogously to its definition for abstract groups: given $\mathfrak{h}$, its *derived subalgebra* $\mathfrak{h}_1$ is defined to be the span of all brackets $[x, y]$ for $x, y \in \mathfrak{h}$. We define $\mathfrak{h}_i$ for $i > 1$ inductively as the derived subalgebra of $\mathfrak{h}_{i-1}$ and say that $\mathfrak{h}$ is solvable if $\mathfrak{h}_i = 0$ for some i.

Given $\mathfrak{g}$ as above, we know that there is a complex algebraic group G, say with maximal torus T and root datum $(X, R, \check{X}, \check{R})$ relative to T, whose Lie algebra is $\mathfrak{g}$. The complexification $\mathbb{C}X$ of the character group X of T then identifies with $\mathfrak{t}^*$, the dual of the subalgebra $\mathfrak{t}$ of $\mathfrak{g}$; the bilinear form $(\cdot, \cdot)$ previously defined on $\mathbb{C}X$ thus induces a form on $\mathfrak{t}^*$, which may be transferred to $\mathfrak{t}$ by identifying a typical coroot $\check{\alpha}$ with $\frac{2\alpha}{(\alpha,\alpha)}$. This transferred form turns out to coincide with the restriction of the Killing form κ of $\mathfrak{g}$ to $\mathfrak{t}$. We also have $\kappa(e_\alpha, e_{-\alpha}) = \frac{2}{(\alpha,\alpha)}$ and it is easy to check from the definition that $\mathbb{C}e_\alpha$ and $\mathbb{C}e_\beta$ are orthogonal under this form whenever $\alpha \neq -\beta$.

The upshot is that *the form* κ *is negative definite when restricted to* $\mathfrak{k}$. As the action of K on $\mathfrak{k}$ (or on $\mathfrak{g}$) clearly preserves this form, we can realize K as a closed subgroup of the orthogonal group $O_n(\mathbb{R})$, where n is the dimension of $\mathfrak{k}$. Hence K is indeed compact as a topological group; this property compensates somewhat for the lack of a root space decomposition of $\mathfrak{k}$. The group K arising in this way from $G = SL_n(\mathbb{C})$ is the *special unitary group* SU_n, that is, the symmetry group of a positive definite Hermitian form on $\mathbb{C}^n$. If $G = SO_n(\mathbb{C})$, then K is the real orthogonal group $SO(n, \mathbb{R})$; if $G = Sp_{2n}(\mathbb{C})$ then $K = Sp_{2n}$, the intersection of $Sp_{2n}(\mathbb{C})$ and the unitary group SU_{2n}.

There are also the *split* real Lie groups, which are exactly the Chevalley groups attached as above to a root system R and the real field. These are the special linear groups $SL_n(\mathbb{R})$ in type A and symplectic groups $Sp_{2n}(\mathbb{R})$ in type C. In types B and D, one more generally has the (disconnected) groups $SO_{p,q}$ and their identity components $SO^0_{p,q}$ for any positive integers p and q, defined to be the symmetry groups of nondegenerate symmetric forms $(\cdot,\cdot)_{p,q}$ of signature (p,q) on $\mathbb{R}^{p+q}$. These last groups are split if and only if either $p = q$ or $p = q \pm 1$.

The groups $SO_{p,q}$ for $|p-q|>1$ are neither compact nor split; in a certain precise sense they are partly compact and partly split. In type A we also have the classical groups $SU_{p,q}$, consisting of matrices in $SL_{p+q}(\mathbb{C})$ preserving a Hermitian form of signature (p,q), and SU^*_{2n}, consisting of matrices in $SL_{2n}(\mathbb{C})$ whose action on $\mathbb{C}^{2n}\cong\mathbb{H}^n$ commutes with right multiplication by the division ring $\mathbb{H}$ of quaternions, where $\mathbb{H}^n$ denotes the space of n-tuples over $\mathbb{H}$. In type C the group $Sp_{2p,2q}$ consists of the matrices in $Sp_{2p+2q}(\mathbb{C})$ preserving a suitable Hermitian form of signature $(2p,2q)$. Finally, in type D, the group SO^*_{2n} consists of matrices in $SO_{2n}(\mathbb{C})$ preserving a suitable skew-Hermitian form on this space. Besides these there are three noncompact real groups of type E_6, two of type E_7, two of type E_8, two of type F_4, and one of type G_2.