Lecture 12-6: The Existence Theorem, part 2

December 6, 2023

We now turn our attention to adjoint groups of non-simply laced types, showing how to construct them from the groups of simply laced type constructed last time.

The missing types, for which we have not yet constructed adjoint groups, are B_n, C_n, F_4, and G_2. We now observe that the Dynkin diagrams D' of types A_n, D_n, and E_6 admit nontrivial *diagram automorphisms* s; that is, nontrivial bijections from the sets of their vertices to themselves such that two vertices are adjacent in the diagram if and only if their images are adjacent. in type A_n, we can reverse the string of vertices, while in type D_n we can interchange the two non-adjacent vertices at one end of the diagram while fixing all the others. In type E_6 we can reverse the string of five vertices forming a subdiagram of type A_5 while fixing the other vertex.

Type D_4 is the most interesting case; here, in addition to the symmetry already observed, we can permute the three outer vertices arbitrarily while fixing the inner one. In all cases note that if one "folds" D' so that simple roots in the same s-orbit lie one on top of the others and single edges connecting simple roots moved by s to simple roots not so moved are combined to make double or triple edges, then we get a Dynkin diagram of type C_n, B_n, B_n, F_4 or G_2 (starting from one of type $A_{2n+1}, A_{2n}, D_{n+1}, E_6$ or D_4), using the unique automorphism of order 2 in the first four cases and either of the two automorphisms of order 3 in the last one. For this reason this operation is called *folding*. To avoid duplication we henceforth ignore the symmetry in type A_{2n}, using this symmetry in type A only if the rank is odd.

Starting with D' and s as above, let R be the root system, and D, R^+ the choices of simple and positive roots in R corresponding to D'. Note that s extends uniquely to an automorphism of X preserving R, D and R^+. We now repeat the construction of the adjoint group G and its maximal torus T corresponding to the datum $\Psi = (X, R, \check{X}, \check{R})$, where X is the $\mathbb{Z}$-span of R, starting with the bilinear form f on X defined last time, but this time arising from the basis of X of simple roots in D, ordered so that simple roots in the same s-orbit occur consecutively in the order.

In this way we ensure that f is s-invariant, in addition to its other properties. Note that s has order 2 or 3, and that if it has order 3 there is just one s-orbit of simple roots of cardinality 3. A simple case-by-case verification shows that in all cases we have

Lemma 10.3.2, p. 181

- $(1-s)X \cap R = \emptyset$;
- If $\alpha, \beta \in R$ and $\alpha - \beta \in (1-s)X$ then α, β lie in the same s-orbit;
- A root is orthogonal to the other roots in its orbit.

This fails in the excluded case of the automorphism of order 2 of a diagram of type A_{2n}. See also the proof of Proposition 10.3.5 below.

Now we observe that the unique **k**-linear map σ on the Lie algebra $\mathfrak{t}$ of a maximal torus of G with $\sigma(1 \otimes \check{\alpha}) = 1 \otimes s.\check{\alpha}, s(e_\alpha) = e_{s\alpha}$ extends to an automorphism of $\mathfrak{g}$, thanks to the construction of $\mathfrak{g}$ and the s-invariance of the form f. We see from the construction of the adjoint group G that σ induces an automorphism of G, also denoted by σ, stabilizing the maximal torus T and having the same order as s. We have $\sigma(u_\alpha(x)) = u_{s\alpha}(x)$ for $\alpha \in R, x \in \mathbf{k}$.

We now show how folding one root datum produces another. Given a root datum $\Psi = (X, R, \check{X}, \check{R})$ with R spanning X over $\mathbb{Z}$ and an automorphism s of the corresponding Dynkin diagram, set $X^s = X/(1-s)X$. Since s permutes the elements of a $\mathbb{Z}$-basis of X, it follows that X^s has no torsion, so is a free abelian group. If $\mathcal{O}$ is an s-orbit in R, let $\alpha_{\mathcal{O}}$ be the coset of α in X^s, for any $\alpha \in \mathcal{O}$; clearly $\alpha_{\mathcal{O}}$ does not depend on the choice of α. Define $\check{\alpha}_{\mathcal{O}}$ to be $m\check{\alpha}$, where m is the cardinality of $\mathcal{O}$. The dual $\check{X}^s$ is the submodule of $\check{X}$ annihilating $(1-s)X$. Then one easily verifies that $\Psi^s = (X^s, R^s, \check{X}^s, \check{R}^s)$ *is a root datum* (Lemma 10.3.4, p. 182); we say that Ψ^s is obtained from Ψ by *folding* according to s.

Denote fixed point sets under an automorphism by a superscript. Then the existence theorem for non-simply-laced root systems follows from

Proposition 10.3.5, p. 183

With notation as above, the connected component $H = (G^\sigma)^0$ is a connected reductive group with maximal torus T^σ. The root datum Ψ^s of $(G^\sigma)^0$ relative to T^σ is obtained as above from the root datum Ψ of G.

Proof.

The character group of the diagonalizable group T^σ is X^σ, a free abelian group, whence T^σ is a torus. We have seen that no root of T in G is trivial on T^σ. It is not difficult to check that the centralizer of T^σ in G is T, which implies that T^σ is a maximal torus in H. If $\alpha_{\mathcal{O}} \in R^s$ set $u_{\alpha_{\mathcal{O}}}(x) = \prod_{\beta \in \mathcal{O}} u_\beta(x)$ for $x \in \mathbf{k}$, where (u_β) is the realization arising in the construction of G. Then $u_{\alpha_{\mathcal{O}}}$ defines an isomorphism of G_a onto a closed subgroup $U_{\alpha_{\mathcal{O}}}$ of H; it is here that the orthogonality of roots in an orbit is crucial, guaranteeing as it does (when combined with the simply laced property of R) that the terms in the product commute pairwise. Arguing as in the proof of Proposition 10.2.8 we see that the $\alpha_{\mathcal{O}}$ are roots of T^σ in H, whence $\dim H \geq |R^s| + \dim T^\sigma$. On the other hand, the Lie algebra $\mathfrak{h} = L(H)$ is a subalgebra of the fixed-point algebra $\mathfrak{g}^\sigma$, whose dimension equals $\dim \mathfrak{t}^\sigma + |R^s|$. Since both $\dim T^\sigma$ and $\dim \mathfrak{t}^\sigma$ equal the number of s-orbits in the Dynkin diagram, we get $\dim H = |R^s| + \dim T^\sigma$ and the proposition follows. □

It turns out that the adjoint group G constructed from a simply laced root system R and the character group X spanned by R coincides with the group of all inner automorphisms of its Lie algebra $\mathfrak{g}$; that is, of the Lie algebra automorphisms of $\mathfrak{g}$ generated by by all exp ad x, as x runs through the elements of $\mathfrak{g}$ such that ad X is nilpotent (or, if the basefield $\mathbf{k}$ is the complex field $\mathbb{C}$, the group generated by all exp ad x as x runs through all elements of $\mathfrak{g}$). Whenever the Dynkin diagram of G admits nontrivial automorphisms, the corresponding automorphisms of $\mathfrak{g}$ (or of G) are never inner; accordingly they are called outer. The full automorphism group of $\mathfrak{g}$ is disconnected; its component group is isomorphic to the group of diagram automorphisms. The existence of an action of the symmetric group S_3 on $G = SO_8(\mathbf{k})$ by automorphisms is a particularly beautiful consequence of the symmetry of the Dynkin diagram of this group that would never have been suspected from its definition. It is called the *principle of triality* and crops up quite often in both the physics and mathematics literature.