Lecture 11-8: Roots and root systems

November 8, 2023

We continue our discussion of roots from last time. Let G be a connected algebraic group with maximal torus T and let $W = N_G(T)/Z_G(T)$ be the Weyl group of T. Also let B be a Borel subgroup of G.

Proposition 7.1.5, p 115

Assume that G is non-solvable and $\dim T = 1$. Then W has order 2 and $\dim G/B = 1$.

Proof.

First note that W has order at most 2, since it acts faithfully by automorphisms on $X = X^*(T) \cong \mathbb{Z}$ and $\mathbb{Z}$ has only two automorphisms. Now fix an isomorphism $\lambda : T \to G_m$. Let B be a Borel subgroup containing T. Let $\phi : G \to GL(V), v \in V$ be a representation and a vector in it with the properties of Theorem 5.5.3 for G and B; we may assume that V is spanned by the images $\phi(x)v$ as x runs through G. Then ϕ defines an isomorphism from G/B onto a closed subvariety $Y = G.\mathbf{k}v$ of $\mathbb{P}V$; we identify G/B with this closed subvariety. Choose a basis $e_1, \ldots, e_n$ of V consisting of weight vectors for the induced representation ρ of G_m on it, so that we have $\rho(a) = a^{m_i} e_i$ for $a \in \mathbf{k}^*, 1 \leq i \leq n$. □

Proof.

(continued) Assume that the indices are arranged so that $m_1 \geq \ldots \geq m_n$. Write a point $x^* \in Y$ in homogeneous coordinates as $(x_1, \ldots, x_n)$. If i, j are respectively the largest and smallest indices with $x_i, x_j \neq 0$, then an easy argument shows that $\mathbf{k}e_i, \mathbf{k}e_j \in Y$, where e_i, e_j are the ith and jth unit coordinate vectors; the morphism $G_m \to \mathbf{P}V$ arising from the action of T extends in two ways to a map from A^1 to $\mathbf{P}V$, with the extra point in its image lying in Y in both cases. You can think of $\mathbf{k}e_i$ as $\lim_{a\to\infty} \phi(a)(\mathbf{k}x)$, while $\mathbf{k}e_j = \lim_{a\to 0} \phi(a)(\mathbf{k}x)$. Running over all the points of Y, we conclude that T has at least two fixed points in G/B, and order of W is exactly 2. □

Proof.

(continued) Letting i, j be the indices arising in the last paragraph, so that $\mathbf{k}e_i, \mathbf{k}e_j$ are fixed points of T in G/B, we find that the points in Y with jth coordinate 0 form a closed T-stable subset Σ which is nonempty. If any component of Σ had dimension at least 1, then the above argument would show that that component would include at least two fixed points of T, together with the one $\mathbf{k}e_j$ that it already has, a contradiction; so Σ is finite. The jth coordinate function f on a suitable open neighborhood of e_i takes the value 0 there and $f^{-1}(0)$ is finite. The second assertion now follows from Corollary 5.2.7, since the dimension of G/B cannot be 0. □

Returning now to the general case (so that the rank of G is arbitrary), let $\alpha \in P'$; we know that $W_\alpha = W(G_\alpha) \subset W$ has order 2. Choose $n_\alpha \in N_{G_\alpha}(T), n \notin Z_{G_\alpha}(T)$ and let s_α be the image of n_α in W. Denote by $\check{X}$ the group $\mathrm{hom}(X, \mathbb{Z})$ of *cocharacters* of $X = X^*G$; this group is also isomorphic to $\mathbb{Z}^n$ and there is a nondegenerate pairing $\langle \cdot, \cdot \rangle$ between X and $\check{X}$. Identify $X, \check{X}$ with subgroups of $V = \mathbb{R} \otimes X, \check{V} = \mathbb{R} \otimes \check{X}$, with $\langle \cdot, \cdot \rangle$ again denoting the pairing between V and $\check{V}$. The action of W on X naturally extends to a linear action on V.

Following p. 116 in the text, we now introduce a positive definite symmetric bilinear form $(\cdot,\cdot)$ on V that is invariant under W: starting with any symmetric positive definite bilinear form f on V, replace $f(x,y)$ by $(x,y) = \sum_{w \in W} f(w.x, w.y)$. The s_α are then reflections in the Euclidean space V: given α, s_α fixes the hyperplane orthogonal to α and sends α to its negative, whence $s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$. Identify $\check{V}$ with the dual of V, or with V itself, using the form $(\cdot,\cdot)$, Using this idenitification set $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)} \in \check{V}$ and call $\check{\alpha}$ the *coroot of* α. Then $s_\alpha(v) = v - \langle v, \check{\alpha}\rangle\alpha, \langle\alpha,\check{\alpha}\rangle = 2$6). The Weyl group W turns out to be generated by the reflections s_α as α runs over P' (Theorem 7.1.9, p. 116).

The upshot is that we can attach to G a *root datum* (7.4.1, p. 124). This consists of the quadruple $(X, R, \check{X}, \check{R})$, where $X, \check{X}$ are free abelian groups of finite rank, in duality by a pairing $\langle \cdot, \cdot \rangle$ taking values in $\mathbb{Z}$, R (denoted earlier by P') and $\check{R}$ are finite subsets of X and $\check{X}$, respectively, equipped with a bijection $\alpha \mapsto \check{\alpha}$ from R to $\check{R}$. Defining $s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha, s_{\check{\alpha}}(y) = y - \langle \alpha, y \rangle \check{\alpha}$ for $\alpha \in R, x \in X, y \in \check{X}$, we have the key properties that $\langle \alpha, \check{\alpha} \rangle = 2$ for $\alpha \in R$ and $s_\alpha R = R, s_{\check{\alpha}} \check{R} = \check{R}$ (properties (RD1) and (RD2) on p. 124). Here of course X is the character group of a maximal torus of G, R the set of its roots, $\check{X}$ the cocharacter group, and the remaining notations are as defined above. The root datum is independent of the choice of maximal torus T since any two such are conjugate.

We identify the root data $(X, R, \check{X}, \check{R}), (Y, S, \check{Y}, \check{S})$ whenever there is a linear isomorphism $\phi : X \to Y$ such that ϕ maps $\check{X}$ onto $\check{Y}$, R onto S, $\check{R}$ onto $\check{S}$, and $\check{\alpha}(\beta) = (\phi\alpha)\check{}(\phi\beta)$ for all $\alpha, \beta \in R$; notice that we are *not* requiring that the positive definite form $(\cdot, \cdot)$ introduced above on $\mathbb{R} \otimes X$ be preserved by ϕ. The root data $(X, R, \check{X}, \check{R})$ arising from algebraic groups have the further property of being *reduced* (see p. 125), meaning that $c\alpha \notin R$ whenever $\alpha \in R$ and $c \in \mathbf{k}$ is different from ± 1; we will prove this later. If for example X and $\check{X}$ both have rank one, so that each of these identifies with $\mathbb{Z}$, then it is easy to check that there are just two reduced root data up to isomorphism, one with roots ± 2 and coroots ± 1, the other with roots ± 1 and coroots ± 2. These data correspond to the groups $SL_2(\mathbf{k})$ and $PSL_2(\mathbf{k})$, respectively.

Letting Q be the subgroup of X generated by R and denoting $\mathbb{R} \otimes Q$ by V, regarded as a Euclidean space equipped with the usual dot product, we see that R satisfies the axioms of a *root system*; that is, (RS1) R is finite, does not contain 0, and spans V; (RS2) if $\alpha \in R$ then the reflection s_α stabilizes R, where $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$; (RS3) If $\alpha \in R$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. (The last condition is sometimes called the *crystallographic condition*.). Root systems arising from reduced root data are also called reduced. The coroot $\check{\alpha}$ is defined to be $\frac{2\alpha}{(\alpha,\alpha)}$ for $\alpha \in R$.

We now return to the salt mines for little longer, proving two more facts about non-solvable groups of rank one; we will then go beyond the text by sketching the classification of root systems and root data, which is purely combinatorial. For now let G be non-solvable of rank one. Fix a Borel subgroup B containing a maximal torus T, let $U = B_u$ be its unipotent radical and let $n \in N_G(T)$ represent the nontrivial element of the Weyl group W, so that $ntn^{-1} = t^{-1}$ for $t \in T$ and $n^2 \in Z_G(T)$.

Lemma 7.2.2, p. 117

- G is the disjoint union of B and UnB.
- $R(G) = (U \cap nUn^{-1})^0$, where $R(G)$ is the solvable radical of G.
- $\dim U/U \cap nUn^{-1} = 1$.

Proof.

Let $x \in G/B$ be the coset B. Then $n.x, x$ are the two distinct fixed points of T in G/B; since $n^{-1}Bn \neq B$ we have $Un.x \neq \{n.x\}$. We have seen that $\dim G/B = 1$; it follows that the complement of $Un.x$ is a finite set S. Since the torus T normalizes U, it must permute this set S, whence it fixes all of its points. It follows that $S \subset \{x, n.x\}$. Since $x \notin Un.x$, $n.x \in Un.x$ we conclude that $Un.x = G/B - \{x\}$; the first assertion follows. □

Proof.

(continued) Since $U \cap nUn^{-1}$ is the isotropy group of $n.x$ in U, part (iii) follows from part (i) and Theorem 5.3.2 (ii). Since the normalizer of a proper closed connected subgroup of a unipotent group always has larger dimension than the subgroup, it must be that $(U \cap nUn^{-1})^0$ is normal in U; since this group is also normalized by T and n, it is normal in G. Part (ii) follows, since $R(G)$ cannot contain a torus. □