Lecture 11-6: More about solvable groups; roots

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Let G be a connected algebraic group.

Theorem 6.4.7, p. 110

Let S be a subtorus of G.

- The centralizer $Z_G(S)$ of S is connected.
- If B is a Borel subgroup of G containing S then $Z_G(S) \cap B$ is a Borel subgroup of $Z_G(S)$; all Borel subgroups of $Z_G(S)$ arise in this way.

Proof.

Set $Z = Z_G(S)$. Take $g \in Z$ and let B be a Borel subgroup containing g. Put $X = \{xB \in G/B : x^{-1}gx \in B\}$. Then X is a closed subvariety of G/B, being a fiber of the projection $Y_1 \to G$ occurring in the proof of Lemma 6.4.4, with $H = B$. As a closed subvariety of G/B, X is complete. Now S acts on X by left multiplication; by the fixed-point theorem there is $xB \in X$ with $x^{-1}Sx \subset B$. Hence there is a Borel subgroup containing both g and S. Then Theorem 6.3.5 (ii) and Corollary 6.3.6 (ii) show that g lies in the identity component Z^0 and part (i) follows. Now let B be as in part (ii). Then $Z \cap B$ is connected by Corollary 6.3.6 (ii) and solvable. To prove the first part of (ii) it suffices to show that $Z/Z \cap B$ is complete. There is a bijective morphism from $Z/Z \cap B$ onto the the Z-orbit $Y = Z.B$ in G/B. Since the map $G \to G/B$ is open, it suffices to show that Y is closed; as the image of $Z \times B$ under a morphism the closure $\overline{Y}$ is irreducible and connected. □

Proof.

(continued) If $y \in Y$ we have $y^{-1}Sy \subset B$; this also holds if $y \in \overline{Y}$. Consider the morphism $\phi : \overline{Y} \times S \to B/B_u$ sending (y, s) to $y^{-1}syB_u$. By the rigidity of diagonalizable groups we conclude that for $y \in \overline{Y}$ we have $y^{-1}sy \in sB_u$, so that $y^{-1}Sy$ is a maximal torus of SB_u. By the conjugacy of maximal tori of that group there is $z \in B_u$ with $y^{-1}Sy = z^{-1}Sz$, so that $y \in Z.B = Y$. Hence Y is closed, as desired; the last assertion in part (ii) follows from the conjugacy of Borel subgroups. □

By contrast with solvable groups, centralizers of semisimple *elements* in general groups need not be connected; see Exercise 6.4.15 (5).

Corollary 6.4.8

Let T be a maximal torus of G. Then $C = Z_G(T)$ is a Cartan subgroup of G and any Borel subgroup of G containing T also contains C.

This follows at once from the theorem with $S = T$, recalling that Cartan subgroups are nilpotent.

Theorem 6.4.9, p. 111

Any Borel subgroup B of G has $N_G(B) = B$.

Proof.

We argue by induction on $\dim G$; the result is trivial if G is solvable. Set $H = N_G(B)$ and let $x \in H$. Fix a maximal torus T of B. Then xTx^{-1} is also a maximal torus of B; by the conjugacy of maximal tori we may assume that $xTx^{-1} = T$. Consider the homomorphism $\psi : t \mapsto xtx^{-1}t^{-1}$ of T onto itself. There are two cases. If the image of ψ is a proper subgroup of T then $S = (\ker \psi)^0$ is a nontrivial torus. Moreover, x lies in $Z = Z_G(S)$ and normalizes the Borel subgroup $Z \cap B$ of Z. If $Z \neq G$ we have $x \in B$ by inductive hypothesis; if $Z = G$ then S lies in the center of G; passing to G/S and again using induction we get $x \in B$. □

Proof.

(continued) Otherwise the image of ψ is all of T. Choose ϕ, V, and v as in the proof of Theorem 5.5.3 for G/B, realizing B as the isotropy subgroup of a line $\mathbf{k}v$ lying inside a rational representation V of G. Then $\phi(B_u), \phi(T)$ fix v, since B_u is unipotent and T lies in the commutator subgroup (H, H), so that ϕ induces a morphism of the complete variety G/B into the affine one V, which must be constant. Then G fixes v, so that $H = G$ and B is normal in G. But then G/B, containing only unipotent elements, is unipotent and G is solvable, forcing $H = G = B$. □

As immediate corollaries we get that *if P is parabolic in G then P is connected and $N_G(P) = P$* and *if P, Q are conjugate parabolic subgroups of G whose intersection contains a Borel subgroup B, then $P = Q$* (Corollaries 6.4.10 and 6.4.11, p. 111). Indeed, P contains a Borel subgroup B, which lies in P^0; if $x \in N_G(P)$ then xBx^{-1} is also a Borel subgroup of P^0, which must be conjugate in P^0 to B, say by y; then $y^{-1}x \in B$ and $x \in P^0$. For Corollary 6.4.11, let $P = xQx^{-1}$. Then B, xBx^{-1} are two Borel subgroups of P, which must be conjugate in P, forcing yx for some $y \in P$ to lie in $N_G(B) = B$ and $x \in P$, so that $P = Q$. We also get

Corollary 6.4.12, p. 111

Let T be a maximal torus of G and B a Borel subgroup containing T. The map $x \mapsto xBx^{-1}$ induces a bijection of $N_G(T)/Z_G(T)$ onto the set of Borel subgroups containing T.

Surjectivity follows from the conjugacy of maximal tori in B; injectivity follows since Borel subgroups are self-normalizing and the normalizer of a torus in a Borel subgroup coincides with its centralizer (Corollary 6.3.6).

We now give a couple of important definitions. The set $\mathcal{B}$ of all Borel subgroups of an algebraic group G is called, naturally enough, its *variety of Borel subgroups*; it may be identified with the homogeneous projective space G/B, where B is any fixed Borel subgroup. Similarly, we have the projective variety $\mathcal{P} = G/P$ of conjugates of a fixed parabolic subgroup P.

If N, N' are normal subgroups of G then $N.N'$ is also normal. Hence there is a unique maximal closed connected normal solvable subgroup of G, called its (solvable) *radical* and denoted $R(G)$. Similarly, there is a unique maximal closed connected normal unipotent subgroup of G, called its *unipotent radical* and denoted $R_u(G)$; we have $R_u(G) = R(G)_u$. We say that G is *semisimple* if $R(G) = e$ and *reductive* if $R_u(G) = e$. The rest of the course will be primarily devoted to the study of reductive algebraic groups.

We now change gears, introducing some combinatorial data attached to a maximal torus T in a connected algebraic group G which play a crucial role in the classification of reductive groups. The dimension n of T is called the *rank* of G (p. 117). This dimension is independent of the choice of T since any two maximal tori are conjugate; the character group X of T is then isomorphic to $\mathbb{Z}^n$. We know that the Lie algebra $\mathfrak{g}$ of G is a rational representation of T via the restriction of the adjoint representation; as such $\mathfrak{g}$ is a direct sum of one-dimensional T-stable subspaces $\mathfrak{g}_\alpha$ called *root spaces*, each corresponding to a character α of T. The nontrivial characters α arising in this way are called *roots* (of T in $\mathfrak{g}$). Denote by P the set of roots. An easy calculation shows that *for any subtorus S of T, the centralizer $Z_G(S) = Z_G(T)$ if and only if S is not contained in any of the subgroups $\ker \alpha$ as α runs over P* (Lemma 7.1.2, p. 114).

For $\alpha \in P$ we denote by G_α the centralizer of the subtorus $\ker \alpha$ of T; this is a closed connected subgroup.

Lemma 7.1.3, p. 114

The G_α generate G as α runs over P; if all G_α are solvable then so is G.

By Corollary 2.2.7 the subgroup H generated by the G_α is closed and connected. Its Lie algebra contains the Lie algebra $\mathfrak{c} = \mathfrak{g}_0$ of the centralizer of T and all root spaces $\mathfrak{g}_\alpha$, whence all of $\mathfrak{g}$, forcing $H = G$. If G_α is solvable then by Theorem 6.4.7 (ii) it lies in some Borel subgroup and thus every Borel subgroup of G; if this holds for all roots α then we have G is a Borel subgroup of itself, so that G is solvable.

We denote by P' the set of roots α such that G_α is non-solvable and by W the quotient $N_G(T)/Z_G(T)$, called the *Weyl group of G* (p. 115). We have seen that W is finite; it acts faithfully as a group of automorphisms of X permuting P and P'. By Corollary 6.4.12 there is a bijection between W and the set of Borel subgroups of G containing T; if B is one such subgroup there is also a bijection between W and the set of T-fixed points in G/B. Fixing $\alpha \in P'$, we note that the group G_α contains $S = \ker \alpha$ in its center and the Weyl group of G_α relative to T coincides with that of G_α/S relative to T/S, where $T/S \cong G_m$ is a one-dimensional torus.