Lecture 11-3: Maximal tori in general algebraic groups

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We begin by proving Theorem 6.3.5, stated last time.

Proof.

We first prove the last assertion. It is clear that $G = T.G_u$, since $T \cap G_u$ is trivial. Now G is a homogeneous space for the group $T \times G_u$ for the action $(t, u).x = txu^{-1}$, with the isotropy group at e being trivial. The tangent map $d\pi_{(e,e)}$ sends $(X, Y) \in L(T) \times L(G_u)$ to $X - Y$ and is bijective, whence it is an isomorphism of varieties. We now prove the other assertions in the case $\dim G_u = 1$. Since G_u is connected it must be isomorphic to G_a. Fix an isomorphism $\phi : G_a \to G_u$ and let $\psi : G \to S = G/G_u$ be the canonical map. We have $\dim S = \dim G - 1$. There is a character α of S such that $g\phi(a)g^{-1} = \phi(\alpha(\psi g)a)$ for $g \in G, a \in \mathbf{k}$. If α is trivial, then G is commutative and the result holds. □

Proof.

(continued) So assume that α is nontrivial. Let $s \in G$ be semisimple and put $Z = Z_g(s)$. By Corollary 5.4.5 we get a direct sum decomposition $\mathfrak{g} = (\mathrm{Ad}(s) - 1)\mathfrak{g} \oplus \mathfrak{z}$. Since $\psi(sxs^{-1}) = \psi(x)$ we have $d\psi \circ (\mathrm{Ad}(s) - 1) = 0$, whence $(\mathrm{Ad}(s) - 1)\mathfrak{g} \subset \ker d\psi = L(G_u)$, with the last equality coming from Corollary 5.5.6 (ii). It follows that $\dim(\mathrm{Ad}(s) - 1)\mathfrak{g} \leq 1$ and $\dim Z^0 = \dim \mathfrak{z} \geq \dim G - 1$. Now assume that $\alpha(\psi s) \neq 1$. Then $Z \cap G_u = e$, whence Z^0 is a closed connected subgroup of G of dimension $\dim G - 1$ with $Z_u^0 = e$; by above results it is a torus. It is maximal and by the last assertion we have $G = Z^0 G_u$. If $g = xy (x \in Z^0, y \in G_u)$ commutes with s then so does y, whence $y = e$; so $Z = Z^0$. We have shown that the centralizer of s is connected if $\alpha(\psi s) \neq 1$. □

Proof.

(continued) If instead $\alpha(\psi s) = 1$ then $L(G_u) \subset \mathfrak{z}$ and we can conclude that Ad s is the identity, whence s lies in the center of G. It then lies in a maximal torus, for example the centralizer of a semisimple element s' with $\alpha(\psi s') \neq 1$; such elements s' exist since we can take the semisimple part of $g \in G$ with $\alpha(\psi g) \neq 1$. It remains to prove the third assertion in the case $\dim G_u = 1$. If T is a maximal torus there is $t \in T$ with $\alpha(\psi t) \neq 1$ and $T = Z_G(t)$. Let T' be another maximal torus and let $t' \in T'$ satisfy $\alpha(\psi t') \neq 1$. Then $T = Z_G(t), T' = Z_G(t')$. Write $t' = t\phi(a), a \in \mathbf{k}$. Then for $b \in \mathbf{k}$ we have $\phi(b)t'\phi(b)^{-1} = t\phi(a + (\alpha(\psi t')^{-1} - 1)b)$. We can take b such that the right side equals t, whence $\phi(b)T'\phi(b)^{-1} = T$, as desired. □

Proof.

(continued) Now consider the general case, taking $\dim G_u > 1$. Let N be as in Lemma 6.3.4. Put $\overline{G} = G/N$. Then $\dim G/G_u = \dim \overline{G}/\dim \overline{G}_u$. Let $s \in G$ be semisimple and let $\bar{s}$ be its image in $\overline{G}$. By induction on $\dim G_u$ we may assume that $\bar{s}$ lies in a maximal torus $\overline{T}$ of $\overline{G}$, whose inverse image H in G is closed, connected, and contains s. Then s lies in a maximal torus of H, which is also one of G. This proves the first assertion; the third one is proved similarly. □

Proof.

(continued) Finally, we have to show that $Z = Z_G(s)$ is connected. Let $G_1 = \{g \in G : sgs^{-1}g^{-1} \in N\}$; this is a closed subgroup containing Z and N (chosen as above) and $G_1/N \cong Z_{\overline{G}}(\bar{s})$. We may assume that $Z_{\overline{G}}(\bar{s})$ is connected, whence G_1 is; if $G_1 \neq G$ then we have by induction on $\dim G$ that Z is connected. Assume now that $G_1 = G$; we may also assume that s is non-central. An argument similar to the one used to prove the second assertion in the case $\dim G_u = 1$ shows that $G = Z^0.N, Z^0 \cap N = e$, whence $Z = Z^0$. This concludes the proof. □

Next we extend the definition of maximal torus to an arbitrary algebraic group and develop the basic properties of such tori. Before doing this we note a consequence of Theorem 6.3.5 (about maximal tori in solvable groups), proved last time.

Corollary 6.3.6, p. 107

Let H be a subgroup of the connected solvable group G whose elements are semisimple.

- H is contained in a maximal torus of G; in particular, any subtorus of H lies in a maximal torus.
- The centralizer $Z_G(H)$ is connected and coincides with the normalizer $N_G(H)$.

Proof.

First of all, H is commutative since the restriction to H of the canonical homomorphism $G \to G/G_u$ is bijective. If H lies in the center of G the result is obvious. Otherwise, take a non-central element s of H. By Theorem 6.3.5 (ii), the centralizer $Z_G(s)$ is connected and contains H. Now the first assertion and the connectedness of $Z_G(H)$ follow by induction on $\dim G$. Finally, if $x \in N_G(H)$, then for $h \in H$ we have $xhx^{-1}h^{-1} \in H \cap (G, G) \subset H \cap G_u = e$, whence the second assertion. □

Now let G be an arbitrary connected algebraic group. We define a *maximal torus* in G to be a subtorus not properly contained in any other subtorus (the obvious definition). A *Cartan subgroup* of G is the identity component of the centralizer of a maximal torus; we will see later that in fact such a centralizer is always connected. For now we observe that *any two maximal tori in G are conjugate* (Theorem 6.4.1, p. 108), since a maximal torus T, being connected and solvable, lies in a Borel subgroup B, with any two maximal tori of B being conjugate. Since any two Borel subgroups of G are conjugate the result follows.

Proposition 6.4.2, p. 108

Let T be a maximal torus of G and $C = Z_G(T)^0$ the corresponding Cartan subgroup.

- C is nilpotent and T is its unique maximal torus.
- There exist elements $t \in T$ lying in only finitely many conjugates of C.

Proof.

Clearly C contains T as a central subgroup. A Borel subgroup B of C containing T also has T has a central subgroup and must be nilpotent, since $B/T \cong B_u$ is nilpotent. By Corollary 6.2.10 we have $C = B$ and by Corollary 6.3.2 (i) T is the only maximal torus of C; this proves (i). For the proof of (ii) we begin by claiming that *for any subtorus S of G there is $s \in S$ with $Z_G(s) = Z_G(S)$* (Lemma 6.4.3, p. 109). To prove this we may assume that $G = GL_n$ and that S consists of diagonal matrices. The diagonal entries define characters of S; let $\chi_1, \ldots, \chi_m$ be the characters so obtained. The elements $s \in S$ with $\chi_i(s) \neq \chi_j(s)$ for $i \neq j$ have the required property and form a dense open subset of S; the claim follows. Now choose $t \in T$ with $Z_G(t) = Z_G(T)$. If t lies in a conjugate gCg^{-1}, then $g^{-1}tg \in T$ and $T \subset Z_G(g^{-1}tg) = g^{-1}Tg$. Since T is maximal it follows that $g \in N_G(T)$. But C is known to have finite index in this last group, whence (ii) follows. □

Lemma 6.4.4, p. 109

Let H be a closed subgroup of G and denote by X the union of the conjugates of H.

- X contains a nonempty open subset of its closure $\overline{X}$. If H is parabolic then X is already closed.
- Assume that H has finite index in its normalizer N and that there exist elements of H lying in only finitely many conjugates of H. Then $\overline{X} = G$.

Proof.

We may assume that H is connected. Then $Y = \{(x, y) \in G \times G : x^{-1}yx \in H\}$ is a closed subset of $G \times G$ isomorphic to $G \times H$ and thus irreducible. If $(x, y) \in Y$ then $(xH, y) \in Y$. It follows that $Y_1 = \{(xH, y) : x^{-1}yx \in H\}$ is an irreducible closed subset of $G/H \times G$. Since $X = \pi Y, \pi$ the second projection, part (i) follows by the definition of parabolic subgroup, since images of morphisms contain nonempty open subsets of their closures. Since the fibers of the projection $Y_1 \rightarrow G/H$ all have dimension $\dim H$ it follows from Theorem 5.1.6 (ii) that $\dim Y_1 = \dim G$. If $x \in H$ lies in only finitely many conjugates of H then $\pi^{-1}x$ is finite, since H has finite index in N. By Theorem 5.2.7 we have $\dim \overline{X} = \dim Y_1 = \dim G$, as desired. □

Theorem 6.4.5, p. 109

- Every element of G lies in a Borel subgroup.
- Every semisimple element of G lies in a maximal torus.
- The union of the Cartan subgroups of G contains a dense open subset.

Let T be a maximal torus, $C = Z_G(T)^0$ the corresponding Cartan subgroup, and B a Borel subgroup containing C (which exists because C is connected and nilpotent). Apply the previous lemma with $H = C$. it follows from Proposition 6.4.2 (i) that $N_G(C) = N_G(T)$. By the rigidity of diagonalizable groups, C has finite index in its normalizer. By Proposition 6.4.2 (ii) the conditions of Lemma 6.4.4 are met; part (iii) follows. Next apply Lemma 6.4.4. (i) with $H = B$; then part (i) follows from (iii). Finally, part (ii) follows from (i) and Theorem 6.3.5 (i).

Corollary 6.4.6., p. 110

Let G be a Borel subgroup of G. Then the center $C(B)$ of B coincides with $C(G)$.

An element in $C(G)$ lies in a Borel subgroup by Theorem 6.4.5 (i), hence in all of them by the conjugacy of Borel subgroups. So $C(G) \subset C(B)$; the reverse inclusion was proved in Corollary 6.2.9.