Lecture 11-29: More about the flag variety

November 29, 2023

Digressing from the text, we say more about the Schubert varieties for GL_n and consider other decompositions of the flag variety into orbits of subgroups other than B.

We first show that (rather surprisingly) it is possible for Schubert varieties to have singular points, even though Schubert cells (of which the varieties are the closures) are affine spaces and thus as nice as possible. Take $G = GL_4(\mathbf{k})$ and consider the Schubert variety V indexed by the permutation $3, 4, 1, 2$. We saw last time that the corresponding Schubert cell C may be thought of as consisting of all matrices of the form

$$\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

To better understand $V = \overline{C}$, we intersect it with the big opposite Schubert cell, that is, the big cell corresponding to the Borel subgroup of *lower* triangular matrices.

This last cell consists of matrices of the form

$$\begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where we are now denoting the coordinate variables by x_{ij}. The defining equations for the intersection are obtained by equating certain subdeterminants of this matrix to 0; here these subdeterminants are the 1×1 determinant corresponding to row 1 and column 4 of the matrix and the 3×3 determinant corresponding to rows $1, 2, 3$ and columns $2, 3, 4$; note that these determinants indeed vanish identically on the relevant submatrices of the first matrix above. Thus the defining equations are $x_{14} = 0$ and $x_{12}(x_{23}x_{34} - x_{24}) - x_{13}x_{34} + x_{14} = 0$.

Forming the Jacobian matrix of these equations at the point where all x_{ij} are 0 and taking the variables in the order $x_{12}, x13, x_{14}, x_{23}, x_{24}, x_{34}$, we get

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

which has rank 1; now the dimension of V is 4 while the dimension of the full flag variety $\mathcal{B}$ is 6, so that the codimension of V is 2. Thus the point in V at which all x_{ij} are 0 is indeed singular. In a similar way, it can be shown that the Schubert variety corresponding to the permutation $4, 2, 3, 1$ is also singular.

Schubert varieties in higher rank groups rapidly become too unwieldy to study by computing Jacobian matrices directly. Remarkably enough, however, these two examples turn out to control all further instances where singular points occur. To make this more precise, we introduce the notion of *pattern avoidance*. Given two permutations $\pi = \pi_1, \ldots, \pi_n$ and $\rho = \rho_1, \ldots, \rho_m$, written in one-line notation with $n \geq m$, we say that π *includes* ρ if there are indices $i_1 < \ldots < i_m$ between 1 and n such that for all j, k with $j < k$ we have $\pi_{i_j} < \pi_{i_k}$ if and only if $\rho_j < \rho_k$. Note that it does not matter whether the indices $i_1, \ldots, i_m$ are consecutive, nor whether the π_{i_j} match the ρ_j. For example, the permutation $3, 5, 7, 1, 4, 2, 6$ includes $1, 4, 2, 3$: taking $i_1, \ldots, i_4$ to be $1, 3, 5, 7$, we find that $\pi_{i_1}, \ldots, \pi_{i_4} = 3, 7, 4, 6$, and the indices $3, 7, 4, 6$ have the same relative order as $1, 4, 2, 3$. We say that π *avoids* ρ if it does not include the latter.

Then a remarkable theorem of Lakshmibai and Sandhya (independently discovered by Wolper, but not stated in the same way) asserts that *the Schubert variety in GL_n indexed by the permutation π of $1, \ldots n$ is singular if and only if π avoids the two permutations* $3, 4, 1, 2$ *and* $4, 2, 3, 1$. This criterion has been extended to types B, C, D (that is, to the orthogonal and symplectic groups) by my colleague Sara Billey, using signed permutations rather than permutations (that is, rearrangements of $1, \ldots n$ in which some or all of the indices may have minus signs attached to them) and longer lists of bad permutations (also called bad patterns) to avoid.

Schubert subvarieties of a general flag variety $\mathcal{B} = G/B$ are closures of B-orbits (Schubert cells) in $\mathcal{B}$. In my own work, I have established similar results for closures of orbits of subgroups other in than B in the flag variety GL_n/B, and for subgroup orbits in the flag varieties of other classical groups. Sticking for simplicity to GL_n for now as the ambient group, it turns out that some especially interesting subgroups arise as follows. First, if $n = p + q$, we have $K = GL_p \times GL_q$ embedded in $G = GL_{p+q}$ in an obvious way as block diagonal matrices, taking the blocks to have sizes $p \times p$ and $q \times q$.

Then K-orbits in G/B turn out to be parametrized by *clans of signature* (p, q), that is, by sequences $c = c_1, \ldots, c_{p+q}$ such that each c_i is either a natural number or a sign $+$ or $-$, every natural number occurs either exactly twice among the c_i or not at all, and finally the number of $+$ signs and pairs of equal numbers among the c_i is exactly p. We identify two clans if they have pairs of equal numbers in the same positions and the same signs in the same position; thus the clans $1, +, 1, -$ and $2, +, 2, -$ are identified, but not $1, +, 1, -$ and $1, +, -, 1$. Thus there are only finitely many clans of a fixed length up to equivalence.

This parametrization arises as follows. There is an obvious decomposition $V = \mathbf{k}^{p+q} = P \oplus Q = \mathbf{k}^p \oplus \mathbf{k}^q$ such that the matrices in K are exactly those preserving P and Q in their natural action on V. Given a complete flag $V_0, \ldots, V_{p+q}$ in $\mathbf{k}^{p+q}$ it lies in the K-orbit $\mathcal{O}_c$ corresponding to the clan c if and only if for each i the number of pairs of equal numbers and $+$ signs occurring among $c_1, \ldots, c_i$ equals the dimension of $V_i \cap P$ and likewise the number of $-$ signs and pairs of equal numbers among $c_1, \ldots, c_i$ equals the dimension of $V_i \cap Q$. In both cases, numbers appearing only once among $c_1, \ldots, c_i$ are not counted. For example, if $c = 1, +, 1, -, -$, so that $p = 2, q = 3, n = 5$, then for any flag (V_i) lying in $\mathcal{O}_c$ the dimensions of the intersections of $V_1, \ldots, V_5$ with P are $0, 1, 2, 2, 2$ and the dimensions of the intersections of these spaces and Q are $0, 0, 1, 2, 3$.

The rule for determining which orbit closures $\overline{\mathcal{O}}_d$ lie in $\overline{\mathcal{O}}_c$ is more complicated to describe than the Bruhat order on S_5, but it has been completely determined. I worked out the criterion for $\overline{\mathcal{O}}_c$ to be smooth in terms of avoidance for clans; I will leave the definitions of inclusion and avoidance for clans to your imagination.

The second interesting subgroup of $GL_n(\mathbf{k})$ to take for K is the orthogonal group $O_n(\mathbf{k})$; recall that this is the group of linear transformations of $\mathbf{k}^n$ preserving the usual dot product $(\cdot, \cdot)$ on $\mathbf{k}^n$. Here the K-orbits are parametrized by involutions, that is, by permutations π such that $\pi_i = j$ for some $j \neq i$ if and only if $\pi_j = i$. The orbit $\mathcal{O}_\pi$ corresponding to the involution π is such that a flag (V_i) lies in $\mathcal{O}_\pi$ if and only if the rank of the dot product on V_i equals the number of indices $j \leq i$ such that $\pi_j \leq i$, for all indices i between 1 and n. The rank of $(\cdot, \cdot)$ on V_i is the dimension of the largest subspace of V_i on which this form is nondegenerate. The criterion for $\overline{\mathcal{O}}_\pi$ to lie in $\mathcal{O}_\mu$ is that $\pi \geq \mu$ (not $\pi \leq \mu$) in the usual Bruhat order. Here again an avoidance criterion for $\overline{\mathcal{O}}_\pi$ to be smooth is known, but the list of bad permutations (all of them involutions) has no fewer than 25 elements.

Finally, the last subgroup K (of GL_n) that I considered is the symplectic group $Sp(2n, \mathbf{k}) \subset GL_{2n}(\mathbf{k})$. Here there is a bilinear nondegenerate skew-symmetric form $(\cdot, \cdot)$ of which K is the group of symmetries. K-orbits in $GL_{2n}(\mathbf{k})/B$ are parametrized by fixed-point-free involutions of $1, \ldots, 2n$, that is, by involutions π such that $\pi_i \neq i$ for all i. The criterion for a flag (V_i) to lie in the orbit $\mathcal{O}_\pi$ corresponding to π is the same as in the orthogonal case; once again the order on K-orbits given by inclusion of their closures is the reverse Bruhat order, so that $\overline{\mathcal{O}}_\pi \subset \overline{\mathcal{O}}_\mu$ if and only if $\pi \geq \mu$. The avoidance criterion for $\overline{\mathcal{O}}_\pi$ to be smooth now involves just 15 bad involutions.