Lecture 11-27: The flag variety

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We use the Bruhat decomposition to produce an analogous decomposition of a very important homogeneous space called the flag variety. Before we do this we wrap up the Bruhat decomposition with a few more results. Throughout G denotes a reductive group with Borel subgroup B and maximal torus T contained in B. The Weyl group of G (relative to T) is denoted by W.

Corollary 8.3.9, p. 145: Bruhat decomposition

An element of G can be uniquely written in the form $u\dot{w}b$ with $w \in W, u \in U_{w^{-1}}, b \in B$.

This follows at once from Bruhat's lemma and Lemma 8.3.6 (ii) (not Lemma 8.3.5 (ii), as indicated in the text).

Corollary 8.3.10, p. 145

The intersection of any two Borel subgroups of G contains a maximal torus.

We may assume that the Borel subgroups are B and $B' = gBg^{-1}$. Writing $g = g\dot{w}b'$ with $w \in W, b, b' \in B$ and letting T be our fixed maximal torus in B, we have $bTb^{-1} \subset B \cap B'$, as claimed. (Note that since Borel subgroups containing a fixed maximal torus T are in bijection to Weyl group elements w, we can express the relation between any two Borel subgroups, sometimes called the *attitude* of one with respect to the other, by such an element.)

A further consequence is

Corollary 8.3.11, p. 146

There is a unique open double coset, namely $C(w_0)$, with w_0 the longest element of W.

This follows since $C(w_0)$ is the only double coset with dimension equal to $\dim G$; it is thus open in its closure G. It is often called the *big cell*. Note that we have now finally identified the analogue of the product $T.\prod_{\alpha\in R} U_\alpha$ of the product $\prod_{\alpha\in R^+} U_\alpha$ occurring in Proposition 8.2.1, where the roots in R are given any fixed order: it is the big cell $C(w_0)$.

Recall now the variety $\mathcal{B}$ of Borel subgroups of G; this is a homogeneous space isomorphic to G/B, since our fixed Borel subgroup B is self-normalizing. In this setting (with G reductive) we give this variety another name, namely the *flag variety* (p. 149). To put this terminology in a specific context, we define a *flag* of a finite-dimensional vector space V over $\mathbf{k}$ to be a chain of subspaces $V_0, V_1, \ldots$ such that each V_i is properly contained in V_{i+1}. The flag is *complete* if this chain is not properly included in a larger one, so that $V_0 = 0, \dim V_i = i$, and the chain ends with $V_n = V$. The Lie-Kolchin Theorem shows at once that an arbitrary Borel subgroup B of $G = GL_n(\mathbf{k})$ is exactly the stabilizer of a complete flag $F = V_0 \subset V_1 \subset \ldots$ in $\mathbf{k}^n$, so that $x \in G$ lies in B if and only if $x.V_i = V_i$ for all i. Moreover, the subgroup B determines and is determined by the flag F. Similarly, if instead F is only a partial (possibly incomplete) flag, then its stabilizer in G is a parabolic subgroup; it turns out that all parabolic subgroups arise in this way.

For general reductive G, let $\pi : G \to G/B$ be the canonical map. For $w \in W$ set $X(w) = \pi C(w)$, where $C(w)$ was defined last time.

Proposition 8.5.1, p. 149

- $\mathcal{B}$ is the disjoint union of the locally closed subvarieties $X(w)$ for $w \in W$. They are the B-orbits in $\mathcal{B}$; in particular there are only finitely many such orbits.
- $X(w)$ is an affine variety isomorphic to $\mathbf{A}^{\ell(w)}$.
- $X(w)$ contains a unique point x_w fixed by T.
- There is a cocharacter λ of T such that for all $x \in X(w)$ we have $\lim_{a \to 0} \lambda(a).x = x_w$; that is, the morphism λ admits a unique extension to $\mathbf{k}$ which maps 0 to x_w.

Proof.

The first assertion is an immediate consequence of the Bruhat decomposition. The second follows from Lemmas 8.3.5 and 8.3.6. The fixed point in part (iii) is just $\pi(\dot{w})$. For the cocharacter in part (iv), choose any λ with $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in R^+$. Then $\lambda(a) u_\alpha(b) \lambda(a)^{-1} = u_\alpha(a^{\langle \alpha, \lambda \rangle} b)$ for $a \in \mathbf{k}^*$, $b \in \mathbf{k}$, from which it follows using Lemma 8.3.5 that for $u \in U_{w^{-1}}$ we have $\lim_{a \to 0} \lambda(a) u \lambda(a)^{-1} = e$, implying the given assertion. □

This result gives a *stratification* of $\mathcal{B}$, that is, a decomposition of this set into locally closed subsets called strata. The strata $X(w)$ are affine spaces, called *Schubert cells* (or *Bruhat cells*, as in the text). The closure $S(w) = \overline{X(w)}$ is called a *Schubert variety*. The open stratum $X(w_0)$ is once again called the big cell; its translates $g.X(w_0)$ cover $\mathcal{B}$.

Lemma 8.5.2, p. 149

The quotient map π has local sections (in the sense defined in section 5.5.7 on p. 95)

This follows at once from Lemma 8.3.6 (ii), covering $\mathcal{B}$ by translates of the big cell as above.

Returning to G, we observe that the closure $\overline{C(w)}$ of $C(w)$ is the union of various B-orbits $C(x)$. We define a partial order on W via $x \leq w$ if $C(x) \subset \overline{C(w)}$. It is immediate that this really is a partial order (called the *Bruhat* or *Bruhat-Chevalley* order) and that we have $x \leq w$ if and only if the Schubert variety $S(x)$ is contained in $S(w)$. There is a beautiful combinatorial description of this order in terms of reduced decompositions in W. Fix one such decomposition $s_1 \ldots s_h$ of $w \in W$ and denote by I_w the set of $x \in W$ that can be written in the form $s_{i_1} \ldots s_{i_m}$ for some indices $1 \leq i_1 < \ldots < i_m \leq h$, or by erasing some factors in the decomposition of w (but preserving the order of the remaining factors). We will see shortly that this definition is independent of the choice of reduced decomposition of w.

Proposition 8.5.5, p. 150

Let $w, x \in W$. Then $x \leq w$ if and only if $x \in I_w$.

Proof.

We first show that *given parabolic subgroups P, Q of G with $P \subset Q$ and a closed subset X of G with $XP = X$, then XQ is closed in G*. Indeed, the image $\overline{X}$ of X in G/P is closed and complete, whence the image of $\overline{X}Q$ in G/P is also complete and must be closed in G/P, whence the preimage XQ of this image is closed in G. Writing $P_i = C(e) \cup C(s_i)$ for $1 \leq i \leq h$, we deduce that $P_1 \ldots P_h$ is an irreducible closed subset of G. By Lemma 8.3.7 it is the union of the double cosets $C(y)$ with $y \in I_w$. Among these there is a unique one of maximal dimension, namely $C(w)$, so $\overline{C(w)}$ is contained in $P_1 \ldots P_h$. Since both sets are irreducible, closed, and have the same dimension, they coincide. In particular, the definition of I_w is indeed independent of the choice of $s_1 \ldots s_h$. □

For $G = GL_n$ there is an elementary way to realize the Bruhat decomposition and define the Bruhat order. Given a complete flag $F = V_0 \subset \ldots \subset V_n$ in $\mathbf{k}^n$, elementary linear algebra shows that there is a basis $v_1, \ldots v_n$ of $\mathbf{k}^n$ such that V_i is the span of $v_1, \ldots, v_i$ for all i. This basis is not unique, but it can be normalized to make it so. First divide the first vector v_i by its rightmost nonzero coordinate, so as to make this rightmost coordinate, say the π_1th, equal to 1. Do the same for the remaining vectors v_i, but in addition subtract a suitable linear combination of $v_1, \ldots, v_{i-1}$ from v_i so as to make the π_jth coordinate of v_i equal to 0 whenever $j < i$ and the last nonzero coordinate of v_i, say the π_ith, has $\pi_i > \pi_j$. The upshot is that the sequence $\pi = \pi_1, \pi_2, \ldots$ of indices is a permutation of $1, \ldots, n$ and the matrix M whose ith row is v_i is in row echelon form; notice that V_i is still the span of $v_1, \ldots, v_i$ and the basis $v_1, \ldots, v_n$ satisfying these conditions is uniquely determined by F.

For example, if $n = 5$ and the permutation $\pi = \pi_1, \ldots, \pi_5$ is $4, 2, 1, 3, 5$, then the matrix M takes the form

$$\begin{bmatrix} * & * & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the $*$s denote arbitrary entries in $\mathbf{k}$. The Schubert cell C indexed by this permutation is thus isomorphic to $\mathbf{k}^4$, with the $*$s furnishing the coordinates of C. In general, the number of $*$s equals the number of *inversions* of π, that is, the number of pairs of indices $i < j$ with $\pi_i > \pi_j$.

Using this realization it is easy to compute the closures of the Schubert cells directly. The upshot is that *given* $\pi = \pi_1, \ldots, \pi_n$ *and* $\rho = \rho_1, \ldots, \rho_n$ *rearrange the sequences* $\pi_1, \ldots, \pi_i$ *and* $\rho_1 \ldots, \rho_i$ *in increasing order as* $\pi'_1, \ldots, \pi'_i$ *and* $\rho'_1, \ldots, \rho'_i$*, for all indices i between 1 and n. Then* $\pi \leq \rho$ *in Bruhat order if and only if* $\pi'_j \leq \rho'_j$ *for all indices i and all* $j \leq i$. For example, the permutations $\pi = 4, 2, 1, 3, 5$ and $\rho = 3, 1, 4, 2, 5$ are not comparable in Bruhat order: rearranging the coordinates of π and ρ as above, we get first 4 and 3 (which shows that if either one is higher in this order it must be π), then $2, 4$ and $1, 3$, and then $1, 2, 4$ and $1, 3, 4$ (which shows that π is not after all higher than ρ).