Lecture 11-22: Bruhat decomposition

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We give a uniform decomposition of any reductive group G using a Borel subgroup B and the Weyl group W relative to a torus T contained in B; this will show among other things that there are only finitely many double cosets BxB in G. We begin with the remaining parts of the lemma partly proved last time. Let R be the root system of G and W its Weyl group, generated by all reflections s_α by roots $\alpha \in R$. Let n be the rank of R, so that R lies in the Euclidean space $\mathbb{R}^n$. Finally, let R^+ be the positive subsystem of R corresponding to B.

Lemma 8.3.2, p. 142, continued

Retain the notation of last time, so that $s_1 \ldots s_h$ is a reduced decomposition of $w \in W$ as a product of simple reflections.

- If α is simple, then $\ell(ws_\alpha) = \ell(w) + 1$ if $w.\alpha \in R^+$ and $\ell(ws_\alpha) = \ell(w) - 1$ if $w.\alpha \in -R^+$.
- If α is simple and $w.\alpha \in -R^+$ then there is a reduced decomposition of w ending with s_α.
- If $s'_1 \ldots s'_h$ is another reduced decomposition of w then there is i with $1 \le i \le h$ and $s_1 \ldots s_{i-1}s_{i+1} \ldots s_h = s'_1 \ldots s'_{h-1}$ (the exchange condition)

Proof.

The first part follows from the formula for $R(ws_\alpha)$ in terms of $R(w)$ given last time. For the next part, observe first that $\ell(w) = \ell(w^{-1}$. Using part (i) of Lemma 8.3.2, proved last time, we get $ws_\alpha = s_1 \ldots s_{i-1}s_{i+1} \ldots s_h$, whence this part follows. Applying this part to $\alpha = s'_h$, we get the last part. □

If s, t are simple reflections with $s \neq t$ denote by $m(s,t)$ the order of the product st in W; from earlier work with root systems of rank 2, we know that $m(s,t) = 2, 3, 4$ or 6. Also note that $m(s,t) = m(t,s)$. It can be shown that *the Weyl group W is presented as an abstract group by the generating set S of simple reflections together with the defining relations $s^2 = 1, sts\ldots = tst\ldots$ ($m(s,t)$ factors on each side) for $s, t \in S, s \neq t$* (Theorem 8.3.4, p. 143; the proof in the text seems incomplete to me). These last relations are called the *braid relations*. A consequence of the proof of this presentation of W is that *given a map ϕ from S into a monoid M such that the $\phi(\alpha)$ satisfy the braid relations, there is a unique extension of ϕ to W such that $\phi(w) = \phi(s_1)\ldots\phi(s_h)$ for any reduced decomposition $s_1 \ldots s_h$ of $w \in W$*.

Recall now that the *Weyl chambers* are the connected components of $\mathbb{R}^n$ with all hyperplanes H_α orthogonal to α removed, for all $\alpha \in R$. We have seen that W acts transitively on the Weyl chambers; in addition, any $w \in W$ sending the *dominant chamber* D to itself (consisting of all $x \in \mathbb{R}^n$ with $(x, \alpha) > 0$ for all $\alpha \in R^+$) necessarily sends roots in R^+ to roots in R^+, so must have length 0, whence $w = 1$. This shows that W acts *simply transitively* on Weyl chambers, so that given any two such chambers C_1, C_2 there is a *unique* $w \in W$ sending C_1 to C_2. In particular, there is a unique $w_0 \in W$ sending D to the *antidominant chamber* $-D$; this is the unique element of largest possible length $|R^+|$. We call it the *long element* of W.

Lemma 8.3.5, p. 144

Let $w \in W$.

- The groups U_α introduced earlier with $\alpha \in R(w)$ generate a closed connected subgroup U_w of $U = B_u$ normalized by T; we have $U_w = \prod_{\alpha \in R(w)} U_\alpha$ (the product being taken in any order).
- The product morphism $U_w \times U_{w_0 w} \to U$ is an isomorphism of varieties.

U_w is closed and connected Corollary 2.2.7 (i) (p. 27) and is clearly normalized by T. Proposition 8.2.3 (p. 138) shows that the product is a group, which then coincides with U_w. Proposition 8.2.1 (p. 137) proves the second part, since $R(w_0 w) = R^+ - R(w)$.

Let $(\dot{w})_{w\in W}$ be a set of representatives in $N_G(T)$ of the elements of W; denote by $C(w)$ the double coset $B\dot{w}B$ (which is easily seen not to depend on the choice of $\dot{w}$). This is an orbit of $B\times B$ acting on G, hence is open in its closure in G.

Lemma 8.3.6, p. 144

Let $w = s_1 \dots s_h$ be a reduced decomposition of $w \in W$; for each index i let α_i be the simple root corresponding to s_i. The morphism $\phi : \mathbb{A}^h \times B \to G$ with $\phi(x_1,\dots,x_h,b) = u_{\alpha_1}(x_1)\dot{s}_1 u_{\alpha_2}(x_2)\dot{s}_2 \dots u_{\alpha_h}(x_h)\dot{s}_h b$ defines an isomorphism $\mathbb{A}^h \times B \cong C(w)$. The map $(u,b) \mapsto u\dot{w}b$ is an isomorphism of varieties from $U_{w^{-1}} \times B$ to $C(w)$.

Proof.

We have $C(w) = B\dot{w}B = U\dot{w}B$. Since elements of W conjugate one-parameter subgroups $(u_i(x))$ of G to one-parameter subgroups we have $\dot{w}^{-1}U_{w_0 s^{-1}}\dot{w} \subset B$, whence $C(w) = U_{w^{-1}}\dot{w}B$. By Lemma 8.3.2 (i) we have $R(w^{-1}) = \{\alpha_1, s_1\alpha_2, \ldots, s_1 \ldots s_{h-1}\alpha_h\}$, whence $U_{w^{-1}} = U_{\alpha_1}(\dot{s}_1 U_{w^{-1}s_1}\dot{s}_1^{-1})$ and $C(w) = U_{\alpha_1}\dot{s}_1 C(s_1 w)$. By induction on h we may assume that the assertion of the lemma holds for $s_1 w = s_2 \ldots s_h$. It follows from the last formula that ϕ is surjective. Then ϕ is the composite of the isomorphism $\mathbb{A}^h \times B \to U_{w^{-1}} \times B$ of the previous result and the morphism $(u, b) \mapsto u\dot{w}b$. That the last morphism is an isomorphism is easily checked by viewing both spaces as homogeneous spaces for $U_{w^{-1}} \times B$ and applying Theorem 5.3.2 (iii) (p. 87). □

One more lemma before we deduce our main result.

Lemma 8.3.7, p. 145

Let $w \in W, s \in S$, where S is the set of simple reflections (relative to a fixed choice of positive roots) Then

- $C(s).C(w) = C(sw)$ if $\ell(sw) = \ell(w) + 1$,
- $C(s).C(w) = C(w) \cup C(sw)$ if $\ell(sw) = \ell(w) - 1$.

Proof.

Let $s = s_\alpha$, α the corresponding simple root. By part (ii) of Lemma 8.3.5 we have $C(s) = U_\alpha \dot{s} B$, whence $C(s).C(w) = U_\alpha \dot{s} C(w)$. For $\ell(sw) = \ell(w) + 1$ the assertion follows from part (i) of this lemma. If $\ell(sw) = \ell(w) - 1$ we have $C(s).C(w) = C(s).C(s).C(sw)$. The lemma follows if we can show that $C(s).C(s) = C(e) \cup C(s)$ Using Lemma 7.2.2 (i) we see that $C(s) \cup C(e)$ is the group G_a of Lemma 7.1.3. By Theorem 7.2.4 the quotient of this group by its radical is isomorphic to SL_2 or PSL_2. The lemma then follows by a direct calculation for these two groups. □

We finally deduce the decomposition we are after.

Theorem 8.3.8, p. 145: Bruhat's lemma

G is the disjoint union of the double cosets $C(w)$ for $w \in W$.

Proof.

Set $G_1 = \cup_{w \in W} C(w)$. From the preceding lemma we deduce that $C(s).G_1 = G_1$ for all $s \in S$. The subgroup of G generated by the maximal torus T and the $U_{\pm\alpha}$ as α runs through S then contains all such $U_{\pm\alpha}$ together with all U_β as β runs through the W-conjugates of simple roots. As these conjugates fill out all of R, it follows from Proposition 8.1.1 that T and the $U_{\pm\alpha}$ generate all of G, whence $G_1 = G$. Now let $w, w' \in W$ and assume that $C(w) \cap C(w') \neq \emptyset$. Since the $C(w)$ are double cosets of B we get $C(w) = C(w')$. Since by Lemma 8.3.6 (i) we have $\dim C(w) = \ell(w) + \dim B$ it follows that $\ell(w) = \ell(w')$; we may assume that $\ell(w) > 0$. By Lemma 8.3.2 there is $s \in S$ with $\ell(sw) = \ell(w) - 1$; by Lemma 8.3.7 we have $C(sw) \subset C(s).C(w') \subset C(w') \cup C(sw')$, whence $C(sw) = C(w')$ or $C(sw) = C(sw')$ since the $C(v)$ are irreducible. Arguing by induction on $\ell(w)$ we get that either $sw = w'$ or $sw = sw'$. The first case is impossible since $\ell(sw) \neq \ell(w')$, whence $w = w'$ and the $C(w)$ are exactly the double cosets of B in G, as desired. □