Lecture 11-20: Reductive groups

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We continue the study of how root data control the structure of the reductive groups that give rise to them. Throughout G is a reductive linear algebraic group with root datum $(X, R, \check{X}, \check{R})$ relative to a maximal torus T and B is a fixed Borel subgroup corresponding to a choice R^+ of positive subsystem of R. Also fix a realization $(u_\alpha : \alpha \in R)$ of R in G. First we prove a general result about solvable groups, from which Proposition 8.2.1 (the last result stated last time) follows.

Lemma 8.2.2, p. 137

Let H be a connected solvable algebraic group with maximal torus S. Assume that there is a set of isomorphisms $v_i(1 \leq i \leq n)$ of G_a onto closed subgroups of H such that there exist nontrivial characters β_i of S, no two of them linearly dependent, with $sv_i(x)s^{-1} = v_i(\beta_i(s)x)$ for $1 \leq i \leq n$ and all $x \in \mathbf{k}$. Also assume that the weight spaces $\mathfrak{h}_{\beta_i}$ are one-dimensional and span $\mathfrak{h}_u = L(H_u)$. Then the morphism $\psi : G_a^n \to H_u$ with $\psi(x_1, \ldots, x_n) = v_1(x_1) \ldots v_n(x_n)$ is an isomorphism of varieties.

Proof.

The proof is by induction on n. If $n = 1$ then H_u equals the image of v_1 (compute dimensions) and the result is trivial. If $n > 1$ let N be a normal subgroup in the center of H_u isomorphic to G_a (see Lemma 6.3.4, p. 105). Then $L(N)$ is an S-stable one-dimensional subspace of $L(H_u)$, which must be one of the weight spaces $\mathfrak{h}_{\beta_j}$. Then Corollary 5.4.7 shows that the centralizer $Z_H(\ker(\beta_j)^0)$ is a group with the properties of H and a one-dimensional unipotent radical (by the linear independence of the β_j). Then N is just the image of v_j. □

Proof.

(continued) For $i \neq j$ let $w_i : G_a \to H/N$ be the homomorphism induced by v_i. We claim that H/N and the w_i satisfy the assumptions of the lemma, relative to the image of S in H/N; this is clear except for the w_i being isomorphisms. Since the images of w_i and w_j overlap trivially (as is easy to check), w_i is injective. Since the weight spaces are one-dimensional the differential dw_i is also injective. By Corollary 5.3.3 (ii), w_i is an isomorphism onto a subgroup of H/N and the claim follows. By induction we may assume that the result holds for H/N; since N is central it easily follows that ψ is bijective. By Lemma 4.4.12 the tangent map $d\psi_{(0,\ldots,0)}$ is bijective. By Theorems 4.3.6 and 5.1.6, ψ is birational. Now Lemma 5.3.4 and Theorem 5.2.8 show that ψ is an isomorphism, as desired. □

Now fix an ordering of all the roots in R extending the previous ordering of $R^+(B)$. It would be natural to expect that an analogue of Proposition 8.2.1 (stated at the end last time) would hold for G and R in place of B and $R^+(B)$. This is not the case; instead the image of the morphism corresponding to ϕ in Proposition 8.2.1 is a proper open subset of G. We will see this later when we prove the Bruhat decomposition (Corollary 8.3.9, p. 145). For now we introduce the *structure constants* that will play a crucial role in presenting a linear algebraic group with specified root datum as an abstract group.

Proposition 8.2.3, p. 138

Fix $\alpha, \beta \in R, \alpha \neq \pm\beta$. There exist constants $c_{\alpha,\beta,i,j} \in \mathbf{k}$ such that the commutator

$$(u_\alpha(x), u_\beta(y)) = \prod_{i\alpha+j\beta\in R, i,j>0} u_{i\alpha+j\beta}(c_{\alpha,\beta,i,j} x^i y^j)$$

for all $x, y \in \mathbf{k}$, where the order of the factors on the right side is the one prescribed by the ordering of R. In particular, if there are no $i, j > 0$ such that $i\alpha + j\beta \in R$, then $u_\alpha(x)$ commutes with $u_\beta(y)$ for all x, y.

Proof.

A simple calculation shows that given α, β there is a positive subsystem of the intersection of R with the subspace W spanned by α and β containing both of these roots, which extends to a positive subsystem of R. Hence we may assume that $\alpha\beta \in R^+$. Then $U_\alpha, U_\beta \in B_u$ and $(u_\alpha(x), u_\beta(y)) = \prod_{\gamma \in R^+} u_\gamma(P_\gamma(x, y))$, where the order of factors in the product is the prescribed one. Conjugating by $t \in T$ we get $P_\gamma(\alpha(t)x, \beta(t)y) = \gamma(t)P_\gamma(x, y)$. Using the linear independence of characters we deduce that $P_\gamma \neq 0$ if and only if $\gamma = i\alpha + j\beta$ for some $i, j \geq 0$. It remains to show that neither i nor j can be 0. Suppose for example that there were a nontrivial factor with $j = 0$; since $i\alpha$ is not a root if $i > 1$ we would have to have $i = 1$. Then the commutator $(u_\alpha(x), u_\beta(y))$ would have a factor $u_\alpha(cx)$ in the product on the right side. Setting $y = 0$ we deduce a contradiction. □

Next we need a property of root systems.

Lemma; cf. Exercise 8.1.12 (3b)

Suppose the Dynkin digram D of the root system R has connected components $D_1, \ldots, D_r$. Then each D_i is the Dynkin diagram of a root system R_i and R is the disjoint union of the R_i, with every root in R_i orthogonal to every root in R_j for $i \neq j$. If D is connected, then R is irreducible in the sense that one cannot partition it into two nonempty subsets with every root in the first subset orthogonal to every one in the second.

Proof.

Let Δ_i be the subset of simple roots corresponding to the nodes of D_i, so that the disjoint union Δ of the Δ_i is the set of simple roots corresponding to D. We know that every root is conjugate by a product of simple reflections a root in Δ_i for some i and that the reflections corresponding to roots in Δ_j fix all linear combinations of roots Δ_i for $j \neq i$. It follows at once that the set of conjugates of a root in Δ_i is exactly the root subsystem R_i of R consisting of roots in the real span V_i of Δ_i and that R is the orthogonal disjoint union of the R_i. If D is connected and R is the orthogonal disjoint union of R_1 and R_2, then either all roots of Δ lie in R_1 or all lie in R_2, by the connectedness. If they all lie in say R_1, then R_2 consists only of roots orthogonal to all roots in Δ; but there are no such roots, so R_2 is empty. □

The consequence of this last result for algebraic groups is

Theorem; cf. Theorem 8.1.5, p. 133

With notations as above, let G be semisimple and let $D_1, \ldots, D_n$ be the irreducible components of the Dynkin diagram D of R, with D_i corresponding to the root system R_i and R the orthogonal disjoint union of the R_i. For each i there is a closed connected normal subgroup G_i of G with root system R_i; we have $(G_i, G_j) = 1$ for $i \neq j$. G is the product of the G_i and the intersection of any G_i and the product of the others is finite. The groups G_i are also quasi-simple in the sense that they have no normal subgroups of positive dimension.

Proof.

For each i let T_i be the subtorus of T generated by the images of the coroots $\check{\alpha}$ for $\alpha \in R_i$. We can then take G_i to be the subgroup generated by T_i and U_α as α runs through the roots in R_i. If $\alpha \in R_k, \beta \in R_\ell$ with $k \neq \ell$, then no combination $i\alpha + j\beta$ is a root for any $i, j > 0$, whence by the above proposition we have $(G_i, G_j) = 1$ for $i \neq j$. Hence the G_i are closed connected normal subgroups and G is their product. The proof of Theorem 8.1.5 in the text shows that each G_i is quasi-simple and each intersects the product of the others in a finite set (since they commute elementwise). □

We conclude with some more combinatorics on the Weyl group W of a root system R. Let Δ be a choice of simple roots.

Proposition; cf. Theorem 8.2.8 (i)

The simple reflections s_α for $\alpha \in \Delta$ generate W.

Given any $\beta \in R$ we know that there is a product w of simple reflections with $w\beta = \alpha \in \Delta$. Then one easily checks that $w^{-1} s_\alpha w = s_\beta$; since the reflections s_β generate W by definition, so do the simple reflections. Hence given any $w \in W$ there is a unique minimum number h such that w is the product of h simple reflections; we denote h by $\ell(w)$ and call it the *length of w* (p. 142). Clearly the identity element is the unique one of length 0, while the simple reflections s_α are the only elements of length 1. If $s_1, \ldots, s_h$ are simple reflections (not necessarily distinct) and $w = s_1 \ldots s_h, \ell(w) = h$, then we call $s_1 \ldots s_h$ a *reduced decomposition* of w; note that a fixed w may have many reduced decompositions.

Given w we can compute the quantity $\ell(w)$ without having to consider any reduced decompositions at all. Fixing a system R^+ of positive roots, set $R(w) = \{\alpha \in R^+ : w.\alpha \in -R^+\}$. Then we have

Lemma 8.3.2, p. 142

Let $s_1 \dots s_h$ be a reduced decomposition of w. Write α_i for the simple root corresponding to the reflections s_i, and recall that the s_i need not be distinct. Then $R(w) = \{\alpha_h, s_h.\alpha_{h-1}, \dots, s_h \dots s_2.\alpha_1\}$, so that in particular $R(w)$ has $h = \ell(w)$ elements.

If $h = 1$, then applying s_h to a positive root β adds or subtracts a multiple of α_h to β, whence $s_h\beta$ is still positive if $\beta \neq \alpha_h$, while $s_h(\alpha_h) = -\alpha_h$, so the result holds. The same reasoning shows that $R(ws_\alpha) = s_\alpha.R(w) \cup \{\alpha\}$ if $w.\alpha \in R^+$, while $R(ws_\alpha) = s_\alpha(R(w) - \{\alpha\})$ if $w.\alpha \in -R^+$. The result follows at once by induction on h. Next time we will begin with more results along these lines.