Lecture 11-17: Root data

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There is a more to a root datum $D = (X, R, \check{X}, \check{R})$ than just the root system R; in this lecture we explore the extent to which different algebraic groups can have the same root system. First note that *the center C of a reductive group G lies in all maximal tori T and coincides with the intersection of the kernels of α as α runs through the weights of T in G,* by an easy argument; in turn *C has positive dimension if and only if the rank of the character group X of T is greater than the rank of R* (Proposition 8.1.8, p. 135)

Given a root datum $(X, R, \check{X}, \check{R})$, the lattice X must contain at least the *root lattice* Q (p. 136), that is, the integral span $\mathbb{Z}R$ of R. If X has the same rank as R, then it must in turn also lie in the *weight lattice* Q, consisting of all $x \in \mathbb{Q}R$ such that $\langle x, \check{R} \rangle \subset \mathbb{Z}$, where $\langle \cdot, \cdot \rangle$ is the canonical pairing between X and $\check{X}$. Since P and Q are free abelian groups of the same finite rank, the quotient P/Q, called the *fundamental group* of R, is finite. The upshot is that *there are only finitely many semisimple algebraic groups up to isomorphism with a fixed root system; more precisely, the isomorphism classes of such groups are in bijection to the lattices between* Q *and* P. *All have isomorphic Lie algebras.*

A calculation shows that the fundamental group of R is cyclic of order n if R is of type A_{n-1}; here P consists of all $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that $\sum a_i = 0$ and $a_i - a_j \in \mathbb{Z}$ for all i, j, while Q consists of the set of such a with $a_i \in \mathbb{Z}$ for all i. A similar calculation shows that the fundamental groups of root systems of types B_n, C_n are both cyclic of order two. In the case of B_n the weight lattice P is the union of $\mathbb{Z}^n$ and the translate $\mathbb{Z}^n + (1/2, \dots, 1/2)$, while Q is just $\mathbb{Z}^n$. In type C_n, P is $\mathbb{Z}^n$ while Q is the sublattice of $\mathbb{Z}^n$ consisting of all vectors whose coordinates sum to an even integer. The case of type D_n is the most interesting one; here the fundamental group is cyclic of order 4 if n is odd but the direct product of two cyclic groups of order 2 if n is even. (To remember which is which, recall that the root systems A_3, D_3 are isomorphic, having the same Dynkin diagram.) In the exceptional cases, only types E_6 and E_7 have nontrivial fundamental groups; these have order 3 and 2, respectively.

We have seen that tori are exactly the algebraic groups G such that the character group $X^*(G)$ is free abelian of rank equal to the dimension of G and whose elements span the coordinate ring $\mathbf{k}[G]$ over $\mathbf{k}$. More precisely, there is an anti-equivalence of categories between tori and free abelian groups of finite rank. Using this fact and taking for granted that given any abstract root datum D there is a reductive group G with root datum D, unique up to isomorphism, we see that for a fixed root system R the inclusion $Q \subset P$ of lattices corresponds to a pair of tori T, T' such that there is a surjective map $T \to T'$ with finite kernel such that $X^*(T) = P, X^*(T') = Q$.

Passing to the algebraic groups G, G' corresponding to the data $(P, R, \check{P}, \check{R}), (Q, R, \check{Q}, \check{R})$, respectively, we deduce that there is a surjective homomorphism from G to G' with finite central kernel. More generally, for any lattice L between Q and P, there is a surjective homomorphism from G to the algebraic group corresponding to $(L, R, \check{L}, \check{R})$ with finite central kernel (whose order equals the index of L in P). The group corresponding to the largest choice P for X is said to be *simply connected*; the group corresponding to the smallest choice Q is said to be *adjoint*, or of *adjoint type*. It is the image $\rho(G) \subset GL(\mathfrak{g})$ of any G with Lie algebra $\mathfrak{g}$ in the adjoint representation ρ of G on $\mathfrak{g}$. Any two such groups G are said to be *isogenous*; a surjective homomorphism from one of them to another with finite kernel is called an *isogeny* (p. 170).

In type A_{n-1} the simply connected group is $SL(n, \mathbf{k})$; the other groups are quotients of this group by a finite central subgroup, which is necessarily cyclic of order dividing n. In particular, the adjoint group is $PSL_n(\mathbf{k}) = SL(n, \mathbf{k})/Z$, where $Z = \langle e^{2\pi i/n} \rangle$ is generated by the scalar matrix $e^{2\pi i/n} I$. In types B and D the adjoint group is $SO_n(\mathbf{k})$; the simply connected one is denoted $Spin(n, \mathbf{k})$. It is a double cover of $SO_n(\mathbf{k})$ and is usually mentioned at some point in the manifolds sequence, begin simply connected in the usual topological sense. (There is also a double cover of the real orthogonal group $SO_n(\mathbb{R})$, denoted $Spin(n, \mathbb{R})$.) In type D_n one also has the adjoint group $PSO(2n, \mathbf{k})$. In type C the simply connected group is $Sp(2n, \mathbf{k})$ and the adjoint one is $PSp(2n, \mathbf{k})$.

We now return to the text, taking up Chapter 8. Let G be a reductive group with root datum $(X, R, \check{X}, \check{R})$, T the maximal torus of G giving rise to this datum. Then the radical $R(G)$ is a central torus (Proposition 7.3.1, p. 120) and the commutator subgroup (G, G) is semisimple, as we will see shortly (Corollary 8.1.6, p. 134).

Proposition 8.1.1, p. 132

- For $\alpha \in R$ there exists an isomorphism u_α from the additive group G_a onto a unique closed subgroup U_α of G such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $t \in T, x \in \mathbf{k}$. We have $\operatorname{Im} du_\alpha = \mathfrak{g}_\alpha$, the α-weight space of T in the Lie algebra $\mathfrak{g}$.
- T and the U_α for $\alpha \in R$ generate G.

If $\alpha \in R$ the group G_α defined previously is reductive and has semisimple rank one, whence the commutator subgroup (G_α, G_α) is semisimple with rank one, and so isomorphic to $SL_2(\mathbf{k})$ or $PSL_2(\mathbf{k})$. A simple calculation in $SL_2(\mathbf{k})$ then gives the first assertion (see also Lemma 7.2.3 (ii)). The second assertion follows since the groups G_α generate G by Lemma 7.1.3.

Corollary 8.1.2, p. 132

The roots of R are the nonzero weights of T in $\mathfrak{g}$; the root spaces $\mathfrak{g}_\alpha$ have dimension one.

Corollary 8.1.3, p. 132

Let B be a Borel subgroup of G containing T and $\alpha \in R$.

- The following are equivalent: (a) $\alpha \in R^+(B)$, the positive subsystem of R corresponding to B; (b) $U_\alpha \subset B$; (c) $\mathfrak{g}_\alpha \subset \mathfrak{b}$.
- $\dim B = r + \frac{1}{2}|R|$, r the rank of G, and $\dim G = r + |R|$

This is a simple calculation, using the previous result.

Lemma 8.1.4, p. 133

- The u_α of Proposition 8.1.1 may be chosen so that for all $\alpha \in R$ the element $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ lies in the normalizer N of T and has image the reflection s_α in the Weyl group W.
- $n_\alpha^2 = \check{\alpha}(-1)$ and $n_{-\alpha} = n_\alpha^{-1}$.
- If $u \in U_\alpha - \{1\}$ there is a unique $u' \in U_{-\alpha} - \{1\}$ such that $uu'u \in N$.
- If $(u'_\alpha : \alpha \in R)$ is a second family with the properties of Proposition 8.1.1 (i) and part (i) above then there are $c_\alpha \in \mathbf{k}^*$ for $\alpha \in R$ with $u'_\alpha(x) = u_\alpha(c_\alpha x), c_\alpha c_{-\alpha} = 1$, for $x \in \mathbf{k}$.

Proof.

We have $U_\alpha \subset (G_\alpha, G_\alpha)$ and $(G_\alpha, G_\alpha) \cong SL_2(\mathbf{k})$ or $PSL_2(\mathbf{k})$ by Theorem 7.2.4. In this way we reduce the proof of part (i) to the case where $G = SL_2(\mathbf{k})$ and T is the diagonal torus. Define the character α of T via $\alpha\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = x^2$. Then a straightforward check shows that we may take $u_\alpha = u_1$, where

$$u_1(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, u_{-\alpha}(x) = n_1 u_\alpha(-x) n_1^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first formula of (ii) follows with $n_\alpha = n_1$. The existence of u' as in part (iii) follows from part (i); part (iv) also follows easily. □

We call a family $(u_\alpha : \alpha \in R)$ with the properties of Proposition 8.1.1 (i) and Lemma 8.1.4 (i) a *realization* of the root system $R = R(G, T)$ in G. Note that the realization determines the coroots $\check{\alpha}$.

Finally we show that the variety structure of a Borel subgroup is as simple as one could hope for. Fix an ordering $(\alpha_1, \ldots, \alpha_m)$ of the positive subsystem $R^+(B)$ corresponding to a Borel subgroup B and a realization (u_α) of $R = \pm R^+(B)$,

Proposition 8.2.1, p. 137

The morphism $\phi : G_a^m \to B_u$ with $\phi(x_1, \ldots, x_m) = u_{\alpha_1}(x_1) \ldots u_{\alpha_m}(x_m)$ is an isomorphism of varieties; in particular, B_u is generated by the groups U_α with $\alpha \in R^+(B)$.

This follows from a more general result, to be proved next time.