Lecture 11-13: More roots and root systems

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Lemma 7.2.3, p. 117

Assume that G is connected semisimple of rank one (so that $R(G) = e$).

- $\dim U = 1, Z_G(T) = T$, and $U \cap nUn^{-1} = e$.
- There is a unique weight α of T in $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum of $\mathfrak{t} = L(T)$ and weight spaces $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$, with $L(U) = \mathfrak{g}_\alpha, L(nUn^{-1}) = \mathfrak{g}_{-\alpha}$.
- The product map $(u, b) \mapsto unb$ is an isomorphism of varieties $U \times B \to UnB = G - B$.

Proof.

We know that $\dim U = 1, \dim B = 2$ and that $U \cap nUn^{-1}$ is finite and unipotent. Since this group is normalized by T it lies in the centralizer of T, which is connected and lies in B. As $\dim B = 2$ we have $Z_G(T) = T$ or $Z_G(T) = B$. The second case would force B to be nilpotent and G to be solvable, which is not the case, so $Z_G(T) = T, U \cap nUn^{-1} = e$, proving part (i). From the classification of one-dimensional groups we know that $U \cong G_a$; let $u : G_a \to U$ be an isomorphism. There is a character α of T such that $tu(a)t^{-1} = u(\alpha(t)a)$ for $a \in \mathbf{k}, t \in T$ and α is nontrivial since $Z_G(T) = T$. If $X \in L(U)$ is a nonzero element in the image of the differential du, then for $t \in T$ we have $\mathrm{Ad}(t)X = \alpha(t)X$, $\mathrm{Ad}(n)X \in L(nUn^{-1})$, and finally $\mathrm{Ad}(t)\mathrm{Ad}(n)X = \alpha(t)^{-1}\mathrm{Ad}(n)X$. □

Proof.

(continued) Now we know that $\dim B \leq 3$; but it follows from the above that $\mathfrak{t} \oplus \mathbf{k}X \oplus \mathbf{k}\mathrm{Ad}(n)X$ is a three-dimensional subspace of $\mathfrak{g}$, so is all of $\mathfrak{g}$, proving part (ii). Finally, we show that $(v, b) \mapsto vb$ is an isomorphism of the variety $nUn^{-1} \times B$ onto $G - nB$, which is equivalent to part (iii). Using part (ii), we check that the tangent map at (e, e) of this map is bijective; the result follows. □

In particular, we deduce, as mentioned above, that *the only multiples of a root* α *that are roots are* $\pm\alpha$, since the subgroup $G_{c\alpha}$ for any $c \in \mathbf{k}^*$ is the same subgroup as G_α, but has only $\pm\alpha$ as the roots occurring in it.

We now classify connected semisimple groups of rank one.

Theorem 7.2.4, p. 118

The only connected semisimple groups of rank one up to isomorphism are $SL_2(\mathbf{k})$ and $PSL_2(\mathbf{k})$.

We have done most of the steps needed to prove this; we refer to pp. 118-20 of the text for the remaining technical details. Note that if $\mathbf{k}$ has characteristic 2, then $SL_2(\mathbf{k})$ and $PSL_2(\mathbf{k})$ are isomorphic as abstract groups but not as linear algebraic groups (their coordinate rings are different).

We now study reduced (crystallographic) root systems in Euclidean space; recall that these are finite subsets R of $V = \mathbb{R}^n$ for some n such that (RS1) R is finite and does not contain 0; (RS2) if $\alpha \in R$, then the only multiples of α in R are $\pm\alpha$; and (RS3) if α, β in R, then $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ and $s_\alpha(\beta) = \beta - ((2\beta, \alpha)/(\alpha, \alpha))\alpha \in R$. Here $(\cdot, \cdot)$ denotes the usual dot product in $\mathbb{R}^n$; recall that $s_\alpha(\beta)$ is just the reflection of β by α, an orthogonal transformation. Denote by W the finite subgroup of $O_n(\mathbb{R})$ generated by the reflections s_α; this is the *Weyl group*. The integer n is called the *rank* of the root system. Two root systems R, R', living in Euclidean spaces V, V' are regarded as isomorphic if there is a linear isomorphism ϕ from V to V' mapping R onto R' such that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\phi\alpha,\phi\beta)}{(\phi\alpha,\phi\alpha)}$ for all $\alpha, \beta \in R$; it is *not* required that ϕ preserve dot products. The *dual* of a root system R, obtained by replacing each $\alpha \in R$ by $\check{\alpha} = 2\alpha/(\alpha, \alpha)$, is easily seen to be a root system. Similarly, given any root datum $D = (X, R, \check{X}, \check{R})$, the quadruple $(\check{X}, \check{R}, X, R)$ is again a root datum, called the *dual* of D (p. 124).

For $\alpha, \beta \in R, \alpha \neq \pm\beta$, we first note that $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} \in \mathbb{N}$; by the Cauchy-Schwarz inequality this last fraction can take only the values $0, 1, 2$, or 3. Hence *the angle between any two nonproportional roots is necessarily a multiple of* $\pi/4$ *or* $\pi/6$; it is this property that makes R crystallographic. Now given any nonproportional $\alpha, \beta \in \mathbb{R}^n$, the corresponding reflections s_α, s_β generate a dihedral subgroup of $O_n(\mathbb{R})$, which is finite if and only if the angle between α, β is a rational multiple of π. For α, β lying in a root system R the only finite dihedral groups that can arise in this way have $4, 6, 8$, or 12 elements, corresponding to the symmetry groups of a pair of orthogonal lines, an equilateral triangle, a square, or a regular hexagon. (Recall that equilateral triangles, squares, and hexagons are the only regular polygons that can tile the plane without overlap.)

In each of these cases, replace V by the span V' of α and β and R by the intersection $R \cap V'$ and this span. It is easy to check that R' is a root system in V'. The vectors α, β may then be taken to lie along axes of symmetry for two orthogonal lines, an equilateral triangle, a square, or a regular hexagon centered at the origin in V'; see p. 154. Replacing α, β by vectors γ, δ also lying along axes of symmetry, but separated as widely as possible, we can arrange that the roots in R' are two points on each of the two lines with all four points equidistant from the origin in the two-line case; of the vertices and $\frac{2}{\sqrt{3}}$ times the midpoints of the sides in the triangle case, the vertices and midpoints of the sides in the square cases, or the vertices and twice the midpoints of the sides in the hexagon case, taking the triangle, square, and hexagon to be centered at the origin. (The reason for the rescaling of the midpoints in the triangle and hexagon cases is to ensure that the ratios $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, and not just the ratios $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$, are integers.)

More precisely, these roots can be taken to be $\pm\gamma, \pm\delta$ (in the two-lines case), $\pm\gamma, \pm\delta, \pm(\gamma+\delta)$ (in the triangle case), $\pm\gamma, \pm\delta \pm (\gamma+\delta), \pm(\gamma+2\delta)$ (square case), or $\pm\gamma, \pm\delta, \pm(\gamma+\delta) \pm (\gamma+2\delta), \pm(\gamma+3\delta), \pm(2\gamma+3\delta)$ (hexagon case). In explicit coordinates, we can take $\gamma = (0,1), \delta = (1,0)$ in the two-line case, $\gamma = (-1/2, \sqrt{3}/2), \delta = (1,0)$ in the triangle case, $\gamma = (-1,1), \delta = (1,0)$ in the square case, and $\gamma = (-\sqrt{2}, 1), \delta = (1,0)$ in the hexagon case. To get cleaner formulas for the coordinates of the roots in the triangle and hexagon cases, it is convenient to replace the ambient vector space $\mathbb{R}^2$ here by the hyperplane H in $\mathbb{R}^3$ consisting of all points (x, y, z) such that $x + y + z = 0$. Then the formulas for γ, δ become $(1,-1,0), (0,1,-1)$ in the triangle case and $(-2,1,1), (1,-1,0)$ in the hexagon case.

To summarize, the root systems of rank 2 up to isomorphism consist of $\pm(1,0), \pm(0,1)$ or of $\pm(1,-1,0), \pm(0,1,-1), \pm(1,0,-1)$ or of $\pm(-1,1), \pm(1,0), \pm(0,1), \pm(1,1)$ or of $\pm(-2,1,1), (1,-1,0), (-1,0,1), (0,-1,1), (1,-2,1), (1,1,-2)$. We say that these four root systems are of *type* $A_1 \times A_1$, *type* A_2, *type* B_2 *(or type* C_2*)*, and *type* G_2, respectively. Clearly any two nonproportional roots in any root system R lie in a subsystem R' of one of these four types.

Notice that in all four cases, the roots labelled γ, δ are such that every root is an integer combination of γ and δ with either all nonpositive or all nonnegative coefficients. We now show that this always happens. Given an arbitrary root system $R \subset V = \mathbb{R}^n$, let U be the complement in V of the union of hyperplanes H_α orthogonal to each root α; an easy exercise shows that U is nonempty. The connected components of U are called *Weyl chambers*. Given x lying in a Weyl chamber, we declare a root $\alpha \in R$ to be *positive* (with respect to x) if $(x, \alpha) > 0$. Then for any root β exactly one of $\pm\beta$ is positive and if β, γ are positive and $\beta + \gamma$ is a root, then it is a positive root. The set R^+ of positive roots is called a *positive subsystem*. Any Weyl chamber C is determined by the positive subsystem of roots corresponding as above to any element of C; the Weyl group W permutes the Weyl chambers. See p. 125 in the text.

Now let C, D be two Weyl chambers and choose $x \in C, y \in D$; let R^+ be the positive subsystem corresponding to C. If $D \neq C$, then there is $\alpha \in R^+$ with $(y, \alpha) < 0$; replacing y by $y' = s_\alpha y = y + c\alpha$ with $c > 0$, we see that $(y', x) > (y, x)$. Thus if we choose z in the W-orbit of y in $\mathbb{R}^n$ with (z, x) maximal, we must have $z \in C$, so that *any two Weyl chambers, or any two positive subsystems, are conjugate under* W. Next, given a positive subsystem R^+ and the corresponding $x \in V$ lying some Weyl chamber, call α indecomposable if it is not the sum of two roots in R^+. Then we claim that *every positive root is a sum of indecomposable positive roots*.

Otherwise, if some positive root is not such a sum, then there is such a root β with (β, x) as small as possible. Then β cannot itself be indecomposable, so write $\beta = \beta_1 + \beta_2$ with the β_i positive. But then the β_i have smaller dot product with x than β does, forcing the β_i to be sums of indecomposable roots, whence β is as well, a contradiction. We now rename the indecomposable positive roots relative to R^+, calling them *simple* (see p. 139). Then (as promised above) every root is an integer combination of simple roots, with all coefficients nonnegative or all coefficients nonpositive; moreover, there is a bijection between positive subsystems and their corresponding sets of simple roots (called simple subsystems), with any two positive subsystems or simple subsystems being conjugate under W.

We conclude by relating positive subsystems to Borel subgroups of linear algebraic groups. We need a simple lemma.

Lemma 7.3.6, p. 122

Let G be a connected semisimple group of rank one, B a Borel subgroup, T a maximal torus of B, and α the unique root of T in B. Let χ be a character of T, regarded as a character of B via the isomorphism $B/B_u \cong T$. Assume that $f \in \mathbf{k}[G]$ is a nonconstant regular function such that for $g \in G, b \in B$ we have $f(gb) = \chi(b)f(g)$. Then $\langle \chi, \check{\alpha} \rangle > 0$.

Proof.

We know that $G \cong SL_2(\mathbf{k})$ or $PSL_2(\mathbf{k})$; since the coordinate ring of the latter is contained in that of the former, we may assume that $G = SL_2(\mathbf{k})$. Then we can take T to be the diagonal matrices in G and B the upper triangular ones. For $x \in G_m$ one checks directly that $\alpha \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = x^2, \check{\alpha}(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. Set $\langle \chi, \check{\alpha} \rangle = a$. By the proof of Theorem 7.2.4, we must have for $z \neq 0$ that $f \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = z^a g(z^{-1})$ for some polynomial g. Regularity at $z = 0$ then forces $a \geq 0$; nonconstancy forces $a > 0$, as desired. □

Now let G be an arbitrary connected algebraic group with Borel subgroup B and maximal torus $T \subset B$. Let α be a root of T in G. Then $G_\alpha \cap B$ is a Borel subgroup of G_α, whence $B' = (G_\alpha \cap B)/(R_u(G_\alpha) \cap B)$ is a Borel subgroup of the reductive group $G' = G + \alpha/R_u(G_\alpha)$, containing the image T' of T. Let $\pm\alpha'$ be the characters of T' corresponding to $\pm\alpha$. It is easy to check that $L(B')$ is the direct sum of $L(T')$ and a one-dimensional weight space, whose weight is either α' or $-\alpha'$. Let $R^+(B)$ be the set of roots obtained in this way as α runs through set $R(G, T)$ of roots of T in G.

Proposition 7.4.6, p. 126

With notation as above, $R^+(B)$ is a positive subsystem.

Proof.

Choose a rational representation $\Phi : G \to GL(A)$ and a nonzero vector $a \in A$ such that B is the stabilizer of the line $\mathbf{k}a$. Then there is a character χ of T, extended to B as above, with $(\phi b).a = \chi(b)a$. Let ℓ be a linear function on A and put $F(g) = \ell((\phi g)a)$ for $g \in G$. Then $F \in \mathbf{k}[G]$ and if $b \in B$ we have $F(gb) = \chi(b)F(g)$. Let α be a root and restrict F to G_α. Since the unipotent radical $R_u(G_\alpha)$ fixes a, the function F is the pullback of a function $F' \in \mathbf{k}[G']$, so that F' has the property of the hypothesis of Proposition 7.3.6. Then we get $\langle \chi, \check{\alpha} \rangle > 0$. If equality holds, then the restriction of F to G_α would be constant for all ℓ and G_α would stabilize a, which is impossible. Hence $\check{R}^+(B)$ is a positive system for $\check{R}$, whence $R^+(B)$ is one for R, as claimed. □

In particular, if α, β are two roots of T in B such that $\alpha + \beta$ is a root, then it is a root of T in B. Also note that since any two positive subsystems are conjugate under the Weyl group W, any positive subsystem takes the form $R^+(B)$ for a unique Borel subgroup B containing the fixed maximal torus T.