Lecture 11-1: Connected solvable groups and maximal tori

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Theorem 6.2.7, p. 102

- A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.
- A Borel subgroup is parabolic.
- Any two Borel subgroups are conjugate.

Proof.

We may assume that G is connected. Let B be a Borel subgroup and P a parabolic subgroup. Applying Borel's fixed point theorem to B and the complete variety G/P we see that P contains a conjugate of B, which is also a Borel subgroup. To finish the proof of the first assertion it suffices to prove the second one. We may assume that G is non-solvable. Then it has a proper parabolic subgroup, which after conjugation we may assume contains B. Then B is Borel in P; by induction on dimension we may assume B is parabolic in P. Hence B is parabolic in G, as desired. Finally, if B, B' are two Borel subgroups, then both are parabolic and each is conjugate to a subgroup of the other, whence they have the same dimension and both are conjugate. □

An easy argument then yields

Corollary 6.2.8, p. 103

Let $\phi : G \to G'$ be a surjective homomorphism of algebraic groups. Let P be a parabolic subgroup (resp. a Borel subgroup) of G. Then ϕP is a subgroup of G' of the same type.

We also get

Corollary 6.2.9, p. 103

If G is connected with center $C(G)$ then $C(G)^0 \subset C(B) \subset C(G)$.

$C(G)^0$ is closed, connected, and commutative, so lies in a Borel subgroup. By the conjugacy of Borel subgroups, it lies in all Borel subgroups, whence the first inclusion. If $g \in C(B)$ the morphism $x \mapsto gxg^{-1}x^{-1}$ induces a morphism $G/B \to G$, which must be constant; the second assertion follows.

Then we have

Corollary 6.2.10, p. 103

If B is nilpotent (as an abstract group) then $G^0 = B$.

A connected nilpotent group contains a nontrivial closed connected subgroup in its center (the subgroup generated by commutators of maximal length). Hence $C(B)$ is nontrivial and central in G; moding out by $C(B)$ and arguing by induction on dimension, the result follows.

We now study connected solvable groups, denoting such a group by G throughout. We will show that G always contains a torus and we will relate its structure to that of the torus. We begin with a famous result.

Theorem 6.3.1, p. 104: Lie-Kolchin Theorem

If G is a closed subgroup of GL_n then there is $x \in GL_n$ with $xGx^{-1} \subset T_n$, the group of upper triangular matrices.

Using induction on n it is enough to prove that the elements of G have a nonzero common eigenvector. This follows from Borel's fixed-point theorem 6.2.6, applied to G acting on $\mathbf{P}^n$.

A more elementary argument is also available. By induction on $\dim G$ we may assume that there is a common eigenvector for the elements of the commutator group (G, G), which is closed and connected. If χ is a character of (G, G) set $V_\chi = \{v \in V : g.v = \chi(g)v, g \in (G, G)\}$. Then G permutes the distinct nonzero spaces V_χ; since G is connected it must in fact stabilize each V_χ. Now it suffices to prove the result when $V = V_\chi$ for some χ. The elements of (G, G) act by scalar multiplications; since these multiplications have determinant one, (G, G) must be finite. Since (G, G) is also connected it must be trivial, forcing G to be abelian. But then we know that G is conjugate to a subgroup of diagonal matrices and the existence of a common eigenvector follows at once.

As an interesting historical aside, Kolchin originally proved this result for differential Galois groups. These are groups attached to the solution spaces of linear homogeneous differential equations with function coefficients coming from a given field of functions, just as ordinary Galois groups are attached to the splitting fields of polynomials with coefficients from a given field of numbers. Differential Galois groups were historically the first groups to be called algebraic.

Corollary 6.3.2, p. 105

Assume further that $G \subset GL_n$ is nilpotent.

- The sets G_s, G_u of semisimple rep. unipotent elements are closed and connected subgroups, with G_s a central torus in G.
- The product map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.

Proof.

We first show that G_s is a central torus. Nilpotence of G forces all n-fold iterated commutators in G to be trivial, for some n. If $s \in G$ is semisimple, the map χ defined in 5.4.1 from conjugation $c(s)$ by s has $\chi(x) = (s, x) = sxs^{-1}x^{-1}$, the commutator of s and x, for all $x \in G$, whence $\chi^n G = e$. By Lemma 4.4.13 we get that $\mathrm{Ad}(s) - 1$ is a linear map on $\mathfrak{g}$ which is both semisimple and nilpotent, so that it is 0. It follows that G_s is closed under multiplication and is indeed a central torus. Now we know that $V = \mathbf{k}^n$ decomposes as a direct sum of one-dimensional subspaces, on each of which G_s acts by scalars. Lie-Kolchin then enables us to put the restriction of G to each subspace in triangular form. The result of the proof proceeds as in the commutative case. □

Corollary 6.3.3, p. 105

Let G be solvable and connected.

- The commutator subgroup (G, G) is closed, connected, unipotent, and normal.
- The set G_u of nilpotent elements is a closed connected nilpotent, and normal subgroup of G; the quotient group G/G_u is a torus.

Proof.

First of all, (G, G) is closed and connected. We may assume that G is a closed subgroup of T_n. Then (G, G) is clearly unipotent. Since $G_u = G \cap U_n$, U_n the upper triangular unipotent matrices, we see that G_u is a closed normal subgroup, nilpotent since U_n is. We have an injective map from G/G_u into the torus $D_n \cong T_n/U_n$. Thus G/G_u is diagonalizable; since it is connected it is a torus. It only remains to show that G_u is connected. Its identity component G_u^0 is normal in G; passing to G/G_u^0, we are reduced to showing that G_u is finite then it is trivial. Now any finite normal subgroup N of a connected linear algebraic group H is central, since for fixed $n \in N$ the morphism from G to N sending g to $g^{-1}ng$ has connected and finite image. Hence $G_u = 1$ and G is nilpotent. The previous result then shows that G_u is connected, as desired. □

Before we prove our next main result we need a lemma.

Lemma 6.3.4, p. 105

Assume that G is not a torus. Then there exists a closed normal subgroup N of G that is isomorphic to G_a and lies in the center of G_u.

Let H a nontrivial closed connected subgroup of G lying in the center of G_u; we have seen that such subgroups exist. If the characteristic p of $\mathbf{k}$ is nonzero we may further assume that $H^p = e$, by replacing H by a suitable power H^{p^e} of itself. Then H is isomoprhic to a vector group G_a^m; if $m = 1$, we are done. Otherwise let $A \subset \mathbf{k}[H]$ be the space of additive functions on H. The torus $T = G/G_u$ acts on H by conjugation; there is a representation of T on $\mathbf{k}[H]$ stabilizing A. We can then find $f \in A$ which is a simultaneous eigenvector for the T-action. Then the identity component $(\ker f)^0$ of the kernel of f has the same properties as H, but has lesser dimension. The lemma then follows by induction.

A *maximal torus* in G is a subtorus with the same dimension as $S = G/G_u$ (p. 106). This turns out to be the same as a subtorus of maximal dimension, as follows from the next result; for now just note that $\dim S$ is indeed the maximal possible dimension of a subtorus.

Theorem 6.3.5, p. 106

- Let $s \in G$ be semisimple. Then s lies in a maximal torus; in particular, maximal tori exist.
- The centralizer $Z_G(s)$ of a semisimple element $s \in G$ is connected.
- Any two maximal tori of G are conjugate.
- If T is a maximal torus, then the product map $\pi : T \times G_u \to G$ is an isomorphism of varieties.

We will prove this next time.