Lecture 10-30: Complete varieties, parabolic and Borel subgroups

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We begin with an account of completeness; as promised earlier, this property is a kind of strengthened form of compactness. The definition (p. 98) is that the variety X is *complete* if for any variety Y the projection morphism $X \times Y \to Y$ is closed, taking closed sets to closed sets. For example, the affine line $\mathbf{A}^1$ is not complete, since the closed subvariety $\{(x, y) : xy = 1\}$ of $\mathbf{A}^1 \times \mathbf{A}^1$ projects to the nonclosed set $\mathbf{A}^1 \setminus \{0\}$.

Proposition 6.1.2, p. 98

If X is complete, then

- A closed subvariety of X is also complete.
- If Y is complete, so is $X \times Y$.
- If $\phi : X \to Y$ is a morphism then ϕX is closed and complete.
- If X is a subvariety of Y, then X is closed.
- If X is irreducible then any regular function defined on X is constant.
- If X is affine then X is finite.

Proof.

The first two assertions follow from the definition. To prove the third assertion, let $\Gamma = \{(x, \phi x) : x \in X\}$ be the graph of ϕ; this is a closed subset of $X \times Y$ (by the separation property of varieties; see Proposition 1.6.11) isomorphic to X, so is complete. Then ϕX is closed because it is the image of Γ under the second projection; it is complete since Γ is. The fourth assertion follows from the third one, applied to the injection $X \to Y$. A regular function on X is a morphism $X \to \mathbf{A}^1$ which induces a morphism $\phi : X \to \mathbf{P}^1$. If X is irreducible and ϕ is nonconstant then it must have infinite range, whence it must be closed and dense in $\mathbf{P}^1$ and so all of $\mathbf{P}^1$, which is impossible. The last assertion follows from the previous one. □

The main result about completeness is

Theorem 6.1.3, p. 98

Any projective variety is complete.

Proof.

If suffices to show that $\mathbf{P}^n$ is complete, so that for any variety Y the projection $\pi : \mathbf{P}^n \times Y \to Y$ is closed. To prove this we may assume that Y is affine and irreducible. Put $A = \mathbf{k}[Y]$, $S = A[T_0, \ldots, T_n]$, and view S as an algebra of functions on $\mathbf{k}^{n+1} \times Y$. If I is a proper homogeneous ideal of S let $\mathcal{V}^*(I)$ be the set of its common zeros in $\mathbf{P}^n \times Y$. We must show that all sets $\pi\mathcal{V}^*(I)$ are closed; for this purpose we may assume that I is prime and that the restriction of π to $\mathcal{V}^*(I)$ is dominant, so that $A \cap I = 0$. Then it is enough to show that for all $y \in Y$ there is $x^* \in \mathbf{P}^n$ with $(x^*, y) \in \mathcal{V}^*(I)$. Let M be the maximal ideal in A of functions vanishing at y. Then $J = MS + I$ is a proper homogeneous ideal of S; we must show that $\mathcal{V}^*(J) \neq \emptyset$. If this fails, then there is $\ell > 0$ such that the set S_ℓ of homogeneous polynomials of degree ℓ lies in J, whence $N = S_\ell / S_\ell \cap I$ is a finitely generated A-module with $N = MN$. Nakayama's Lemma then shows that there is $a \equiv 1 \mod M$ with $aN = 0$, $aS_\ell \subset I$. Since $a \notin I$ we have $S_\ell \subset I$, $N = 0$, forcing $\mathcal{V}^*(I) = \emptyset$, which is absurd. Hence $\mathcal{V}^*(J) \neq \emptyset$, as desired. □

An easy argument shows that *a connected algebraic group that is complete as a variety is necessarily commutative* (Theorem 6.1.6, p. 100). Such groups are called *abelian varieties*; the elliptic curves mentioned earlier are examples. These are very rich objects fully deserving a course of their own. For now we note that this is the last time we will do pure algebraic geometry in the course; henceforth all results will pertain directly to groups.

Now we have an easy lemma. Let G be an algebraic group.

Lemma 6.2.1, p. 101

Let X, Y be homogeneous spaces for G and $\phi : X \to Y$ a bijective G-morphism. Then X is complete if and only if Y is.

By Theorem 5.3.2, for any variety Z the map $(\phi,\mathrm{id}) : X \times Z \to Y \times Z$ is a homeomorphism of topological spaces. The lemma follows at once.

We say that a subgroup P of G is *parabolic* if the quotient variety G/P is complete. Although it has been almost sixty years since Borel and Tits introduced this terminology, no one seems to know why they chose this word.

Lemma 6.2.2, p. 101

If P is parabolic in G then G/P is projective.

The construction of quotient spaces shows that G/P is quasi-projective, whence by Proposition 6.1.2 it is projective.

Lemma 6.2.3, p. 101

Let P be parabolic in G and Q parabolic in P; then Q is parabolic in G.

Proof.

We must show that for any variety X the projections map $G/Q \times X \to X$ is closed. This reduces by Theorem 5.3.2 to showing that for any closed set $A \subset G \times X$ such that $(g, x) \in A$ if and only $(gq, x) \in A$ for all $q \in Q$, the projection A' of A is closed. Consider the morphism $\alpha : P \times G \times X \to G \times X$ sending (p, g, x) to (gp, x). If A is as above then $\alpha^{-1}A = \{(p, g, x) : (gp, x) \in A\}$, which is closed in $P \times G \times X$. Completeness of P/Q implies that the projection of this set to $G \times X$, that is, the union $\cup_{(g,x)\in A}(gP, x)$, is closed. The completeness of G/P then shows that the projection of this set to X is closed. Since the projection is A' the lemma follows. □

Lemma 6.2.4, p. 101

Let P be parabolic in G. If Q is a closed subgroup containing P then Q is also parabolic. Also P is parabolic in G if and only if P^0 is parabolic in G^0.

The first assertion follows from Proposition 6.1.2, as G/Q is the image of G/P under a morphism. To prove the second assertion note first that G^0 is parabolic in G. If P is parabolic in G then P^0 is parabolic in G^0 by the previous lemma, and also in G^0,since G^0/P^0 is closed in G/P^0. If conversely P^0 is parabolic in G^0 then it is parabolic in G by the previous lemma and P is parabolic in G by the first assertion.

Proposition 6.2.5, p. 102

A connected group G has proper parabolic subgroups if and only if it is not solvable (as an abstract group).

We may assume that G is closed in some $GL(V)$. Then G acts on the projective space $\mathbf{P}V$. Let X be a closed orbit for this action. Then X is projective and thus complete. Choose $x \in X$ and let P be its isotropy subgroup. Then $gP \mapsto g.x$ defines a bijective morphism of homogeneous spaces, whence P is parabolic in G. If $P = G$ put $V_1 = V/x$. Then G acts on $\mathbf{P}V_1$. There is a closed orbit for this action, whence a parabolic subgroup P_1. Continuing in this way, we either get a proper parabolic subgroup or realize G as a subgroup of T_n, the upper triangular matrices. It is easy to see that any such subgroup is solvable.

To finish the proof we have to show conversely that if G is connected and solvable then it has no proper parabolic subgroups. Assume contrarily that P is one of maximal dimension; we may assume it is connected. The commutator subgroup (G, G) is closed and connected and $Q = P.(G, G)$ is a connected parabolic subgroup containing P, forcing $Q = P$ or $Q = G$. In the first case we get a bijection of homogeneous spaces $(G, G)/(G, G) \cap P \to G/P$, whence $(G, G) \cap P$ is parabolic in (G, G). By induction on dimension we may assume that $(G, G) \cap P = (G, G)$, i.e. that $(G, G) \subset P$, contradicting the assumption that $Q = G$. Finally, if $Q = P$, then again $(G, G) \subset P$, forcing P to be normal and G/P to be affine and finite, a contradiction.

Theorem 6.2.6, p. 102: Borel's fixed point theorem

Let G be a connected solvable linear algebraic group and X a complete G-variety. There is a point in X fixed by all $g \in G$.

Indeed, G has a closed orbit in X. The isotropy group of a point of it is parabolic, whence this group must be all of G, as desired.

A *Borel subgroup* of G is a closed connected solvable subgroup which is maximal for these properties. Such subgroups exist (take one of maximal dimension).