# Lecture 10-27: Homogeneous spaces

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Given a closed subgroup $H$ of an algebraic group $G$, our final goal for the chapter is to give the structure of a quasi-projective variety to the quotient $G/H$.

We begin by noting that *every irreducible component of a homogeneous space* $X$ *for* $G$ *is a homogeneous space for its identity component* $G^0$ *and* $X$ *is the disjoint union of its irreducible components* (Lemma 5.3.1, p. 86). This follows at once by the same argument showing that $G$ is the disjoint union of its irreducible components. Next we have

## Theorem 5.3.2, p. 86

Let $G$ be an algebraic group and $\Phi : X \to Y$ a $G$-equivariant morphism of $G$-homogeneous spaces. Set $r = \dim X - \dim Y$.

- For any variety $Z$ the morphism $(\phi,\text{id}) : X \times Z \to Y \times Z$ is open.
- If $Y'$ is an irreducible closed subvariety of $Y$ and $X'$ an irreducible component of $\phi^{-1}Y'$ then $\dim X' = \dim Y' + r$. In particular, if $y \in Y$ than all irreducible components of $\phi^{-1}y$ have dimension $r$.
- $\phi$ is an isomorphism if and only if it is bijective and for some $x \in X$ the tangent map $d\phi_x : T_xX \to T_{\phi x}Y$ is bijective.

**Proof.**

Using the previous result we reduce to the case that $G$ is connected and $X$, $Y$ are irreducible. Then $\phi$ is surjective and thus dominant. Let $U \subset X$ be an open subset with the generic properties proved earlier for dominant morphisms. Then all translates $g.U$ have the same properties; since these cover $X$ we get the first two assertions. If $\phi$ is bijective we know that $\mathbf{k}(X)$ is a purely inseparable extension of $\mathbf{k}(Y)$. If $d\phi_x$ is surjective for some $x$ we see from Theorem 4.3.7 that this extension is also separable. Hence $\mathbf{k}(X) = \mathbf{k}(Y)$ and $\phi$ is birational, and thus an isomorphism on an open subset. Covering $X$ by translates of this set we see that $\phi$ is an isomorphism. □

The simplest example of a bijective homomorphism of algebraic groups which fails to be an isomorphism of algebraic groups occurs (as mentioned before) with $G = G_m$ or $G_a, \phi : G \to G, \phi(x) = x^p$. Here the differential $d\phi$ is the zero map at every point.

## Corollary 5.3.3, p. 87

If $G \to G'$ is a surjective homomorphism of algebraic groups, then $\dim G = \dim G' + \dim \ker \phi$ and $\phi$ is an isomorphism if and only if both $\phi$ and the tangent map $d\phi_e$ are bijective.

This is clear.

Now let $G$ be a connected algebraic group and let $\sigma$ be an automorphism of $G$. Set $G_\sigma = \{x \in G : \sigma x = x\}$; this *fixed subgroup* is a closed subgroup of $G$. Denote by $\chi$ the morphism sending $x \in G$ to $(\sigma x)x^{-1}$. The differential $d\sigma$ is an automorphism of the Lie algebra $\mathfrak{g}$ of $G$. Setting $\mathfrak{g}_\sigma = \{X \in \mathfrak{g} : d\sigma(X) = X\}$, we get $d\chi(L(G_\sigma) = 0$, whence $L(G_\sigma) \subset \mathfrak{g}_\sigma = \ker d\chi$, since $d\chi = d\sigma - 1$ by Lemma 4.4.13. In general, equality does not hold; the simplest example occurs in Exercise 5.4.9 (1) on p. 90 of the text. It is easy to check that in fact $L(G_\sigma) = \mathfrak{g}_\sigma$ *if and only if* $\chi$ *is separable when viewed as a morphism from* $G$ *to* $\overline{\chi G}$ (Lemma 5.4.2, p. 88).

## Theorem 5.4.4, p. 89

Let $\sigma$ be a semisimple automorphism of $G$ (so that $\sigma^*$ acts semisimply on $\mathbf{k}[G]$). With notation as above, the image $\chi G$ is closed and $\chi$ is separable when viewed as a morphism from $G$ to $\chi G$. We also have $L(G_\sigma) = \mathfrak{g}_\sigma$.

**Proof.**

We may assume that $G$ is a closed subgroup of $GL_n$ and that $\sigma x = sxs^{-1}$ for some semisimple element of $GL_n$, which we may take to be a diagonal matrix. If $G = GL_n$ then the last assertion is easy. If $G$ is arbitrary, then regard $\sigma$, which is conjugation by $s$, as an automorphism of $GL_n$, and extend $\chi$ similarly. Let $X \in T_e\overline{\chi G}$. Since $T_e\overline{\chi G} \subset T_e\overline{\chi(GL_n)}$ and since we already know the result for $GL_n$ there is $Y \in \mathfrak{gl}_n$ with $X = d\sigma(Y) - Y$. Semisimplicity of $s$ implies that $d\sigma$ is a semisimple automorphism of $\mathfrak{gl}_n$, stabilizing the subspace $\mathfrak{g}$. Then this subspace has a $d\sigma$-stable complement. Hence we may take $Y \in \mathfrak{g}$ and $d\chi$ is surjective, when $\chi$ is separable by Theorem 4.3.6; also the second assertion holds. □

## Proof.

(continued) It remains to show that $\chi G$ is closed. Set $m(T) = \prod_{i=1}^{r}(T - a_i)$, where the $a_i$ are the distinct eigenvalues of $s$. Let $S \subset GL_n$ consist of the matrices $x$ such that $x$ normalizes $G$, $m(x) = 0$, and the characteristic polynomial of the restriction of Ad $x$ to $\mathfrak{g}$ equals that of $d\sigma$. Then $S$ is closed and contains $s$. Since $m$ has distinct roots, all elements of $S$ are semisimple. Now for $x \in S$ put $G_x = \{g \in G : gxg^{-1} = x\}$ and $\mathfrak{g}_x = \{X \in \mathfrak{g} : \mathrm{Ad}(x)X = X\}$. Then we have shown that $\dim G_x = \dim \mathfrak{g}_x$. But the latter dimension equals the multiplicity of the eigenvalue 1 of the restriction of Ad $(x)$ to $\mathfrak{g}$, which equals $\dim \mathfrak{g}_\sigma$. Hence $\dim G_x = \dim G_\sigma$ for all $x \in G$. Now $G$ acts on $S$ by inner automorphisms and by Theorem 5.3.2 all orbits have dimension $\dim G = \dim G_\sigma$. But then by Lemma 2.3.3 all orbits are closed; since $\chi G$ is a translate of an orbit, the theorem is proved. It also follows that *the conjugacy class* $C = \{xsx^{-1} : x \in G\}$ *is closed and if* $Z = G^s$ *is the centralizer of* $s$ *in* $G$*, then* $\mathfrak{g} = (Ad(s) - 1)\mathfrak{g} \oplus L(Z)$ (Corollary 5.4.5, p. 89). □

In general, conjugacy classes in $G$ are not closed; for example, the unique nontrivial conjugacy class of a unipotent element in $SL_2$ has the identity element in its closure; in fact, all conjugacy classes of unipotent elements in an a reductive algebraic group (one admitting no nontrivial unipotent normal subgroup) have the identity element in their closures.

Now we are ready to tackle the quotient group construction. Let $H$ be a closed subgroup of the linear algebraic group $G$.

**Lemma 5.5.1, p. 91**

There is a finite dimensional subspace $V$ of $\mathbf{k}[G]$ and subspace $W$ of $V$ such that

- $V$ is stable under right translations by elements of $G$.
- $H = \{x \in G : \rho(x)W = W\}, \mathfrak{h} = \{X \in \mathfrak{g} : X.W \subset W\}$.

**Proof.**

Let $I \subset \mathbf{k}[G]$ be the ideal of functions vanishing on $H$ and let $V$ be a finite-dimensional $\rho(G)$-stable subspace of $\mathbf{k}[G]$ containing a set $f_1, \ldots, f_r$ of generators of $I$. Set $W = V \cap I$. Then I claim that $W$ has the required properties. Indeed, if $x \in H$ then $\rho(x)W = W$, by an easy calculation. Conversely, if $\rho(x)W = W$, then $\rho(x)I \subset I$ and $x \in H$ by Lemma 2.3.8. The proof of the corresponding Lie algebra property is similar, using Lemma 4.4.7. Note that if $\phi$ is the rational representation of $G$ in $V$ defined by $\rho$, then we have $d\phi(X).f = X.f$ for $f \in V, X \in \mathfrak{g}$. □

Now let $V$ be an arbitrary finite-dimensional vector space and $W$ a subspace of dimension $d$. The $d$th exterior power $\wedge^d V$ of $V$ contains the one-dimensional subspace $L = \wedge^d W$. Let $\phi$ be the canonical representation of $GL(V)$ on $\wedge^d V$; the actions of $GL(V)$ and its Lie algebra on this space were described last week.

## Lemma 5.5.2, p. 92

For $x \in GL(V)$ we have $x.W = W$ if and only if $(\phi x)(L) = L$, while for $X \in \mathfrak{gl}(V)$ we have $X.W \subset W$ if and only if $(d\phi X)(L) \subset L$.

The only if assertions are clear. Choose a basis $(v_1, \ldots, v_n)$ of $V$ such that $(v_1, \ldots, v_d)$ is one of $W$. Given $x \in GL(V)$, we may choose the $v_i$ such that $(v_{\ell+1}, \ldots, v_{\ell+d})$ is a basis of $x.W$. If $x.L = L$ this is clearly impossible unless $\ell = 0$. Similarly, if $X \in \mathfrak{gl}(V)$, and $e = v_1 \wedge \ldots \wedge v_d$ then $(d\phi X)(e) = \sum_{i=1}^{d} v_1 \wedge \ldots \wedge Xv_i \wedge \ldots \wedge v_d$. Writing $Xv_i = \sum a_{ij} v_j$ it follows that $(d\phi X)e = \sum_{i=1}^{d} \sum_j a_{ij} v_1 \wedge \ldots \wedge v_j \wedge \ldots \wedge v_d$. Then if $a_{ij} \neq 0$ for some $i \leq d, j > d$, then the subspace $L$ is not mapped into itself by $(d\phi X)$, proving the second assertion.

The lemmas now yield the following result, with $G$ and $H$ as before.

**Theorem 5.5.3, p. 92**

There is a rational representation $\phi : G \to GL(V)$ and a nonzero $v \in V$ with $H = \{x \in G : (\phi x)v \in \mathbf{k}v\}, \mathfrak{h} = \{X \in \mathfrak{g} : (d\phi X)v \in \mathbf{k}v\}$.

As a corollary we get

## Corollary 5.5.4, p. 93

There is a quasi-projective homogeneous space $X$ for $G$ together with a point $x \in X$ such that the isotropy group of $x$ in $G$ is $H$, the morphisms $\psi : g \mapsto g.x$ of $G$ to $X$ defines a separable morphism $G^0 \to \psi G^0$, and the fibers of $\psi$ are the cosets $gH$ of $H$.

. Let $V, v$ be as in the theorem and let $x$ be the point in projective space $\mathcal{P}V$ defined by the line $\mathbf{k}v$. Denote by $\pi : V \setminus \{0\} \to \mathcal{P}V$ the map sending a vector to the line passing through it. Now $G$ acts in an obvious way on $\mathcal{P}V$. Denoting by $X$ the $G$-orbit of $x$, the assertions follow from the theorem and Theorem 4.3.7 (ii).

Now we finally construct the quotient space $G/H$. Of course its points are the cosets $gH$; let $\pi : G \to G/H$ be the canonical map. Define $U \subset G/H$ to be open if and only if $\pi^{-1}U$ is open in $G$. This defines a topology on $G/H$ such that $\pi$ is an open map. Define a sheaf $\mathcal{O}$ of **k**-valued functions on $G/H$ by declaring that if $U \subset G/H$ is open then $\mathcal{O}(U)$ is the ring of functions $f$ on $U$ such that $f \circ \pi$ is regular on $\pi^{-1}U$. It is easy to check that this indeed defines a sheaf of functions. Now, defining $X$ as in the previous corollary, one can show that the map $\phi : G/H \to X$ sending $gH$ to $g.x$ is an isomorphism. The details are tricky, requiring Zariski's main theorem. I refer to p. 94 of the text. As a corollary one gets that *$G/H$ is a quasi-projective variety of dimension* $\dim G - \dim H$ and *if $G$ is connected then the canonical morphism $\pi : G \to G/H$ is separable* (Corollary 5.5.6, p. 94). In general, $G/H$ does not have the structure of an affine variety, but *it does have this structure if $H$ is normal in $G$ and then $G/H$ is a linear algebraic group* (Proposition 5.5.10, p. 96).

More generally, let $H$ be a closed subgroup of the linear algebraic group $G$ and $X$ an irreducible $H$-variety on which $H$ acts on the right. Assume that the morphism $\pi : G \to G/H$ *has local sections* in the sense that $G/H$ is covered by open sets $U$ such that $U$ admits a morphism $\sigma : U \to G$ with $\pi \circ \sigma = \mathrm{id}_U$. Then *a quotient* $(G \times X)/H$ *exists; it is the fibered product* $G \times^H X$ *constructed by starting with* $G \times X$ *and then identifying any point* $(g, x)$ *with* $(gh, h^{-1}.x)$ (Lemma 5.5.8, p. 95).